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Vol. 6

S(n) is the smallest integer such that S(n) is divisible by n

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On some numerical functions

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In this paper we prove that the following numerical functions:

1. $F_S: N^* \to N$, $F_S(x) = \sum_{i=1}^{\pi(x)} S(p_i^x)$, where p_i are the prime natural numbers which are not greater than x and $\pi(x)$ is the number of them,
2. $\theta: N^* \to N$, $\theta(x) = \sum_{p_i \mid x} S(p_i^x)$, where p_i are the prime natural numbers which divide x,
3. $\tilde{\theta}: N^* \to N$, $\tilde{\theta}(x) = \sum_{p_i \nmid x} S(p_i^x)$, where $p_i$are the prime natural numbers which are smaller than x and do not divide x,

which involve the Smarandache function, does not verify the Lipschitz condition. These results are useful to study the behaviour of the numerical functions considered above.

Proposition 1 *The function* $F_S: N^* \to N$, $F_S(x) = \sum_{i=1}^{\pi(x)} S(p_i^x)$, *where* p_i *and* $\pi(x)$ *have the signifience from above, does not verify the Lipschitz condition.*

Proof. Let $K > 0$ be a given real number, $x = p$ be a prime natural number, which verify $p > \left[\sqrt{K} + 1\right]$ and $y = p - 1$. It is easy to see that $\pi(p) = \pi(p-1) + 1$, for every prime natural number p, since the prime natural numbers which are not greater than p are the same as those of $(p-1)$ in addition to p. We have:

$$|F_S(x) - F_S(y)| = F_S(p) - F_S(p-1) =$$

$$= \left[S(p_1^p) + S(p_2^p) + \cdots + S(p_{\pi(p-1)}^p) + S(p^p)\right] -$$

$$- \left[S(p_1^{p-1}) + S(p_2^{p-1}) + \cdots + S(p_{\pi(p-1)}^{p-1})\right] =$$

$$= \left[S(p_1^p) - S(p_1^{p-1})\right] + \cdots + \left[S(p_{\pi(p-1)}^p) - S(p_{\pi(p-1)}^{p-1})\right] + S(p^p).$$

But $S(p_i^p) \geq S(p_i^{p-1})$ for every $i \in \overline{1, \pi(p-1)}$, therefore we have

$$|F_S(x) - F_S(y)| \geq S(p^p).$$

Because $S(p^p) = p^2$, for every prime p, it follows:

$$|F_S(x) - F_S(y)| \geq S(p^p) = p^2 > K = K \cdot 1 = K\,(p - (p-1)) = K\,|x - y|\,.$$

We have proved that for every real $K > 0$ there exist the natural numbers $x = p$ and $y = p - 1$, chosen as above, so that $|F_S(x) - F_S(y)| > K\,|x - y|$, therefore F_S does not verify the Lipschitz condition.

Remark 1 *Another proof, longer and more technical, can be made using a rezult which asserts that the Smarandache function S also does not verify the Lipschitz condition. We have chosen this proof because it is more simple and free of another results.*

Proposition 2 *The function $\theta : N^* \to N$, $\theta(x) = \sum\limits_{p_i | x} S(p_i^x)$, where p_i are the prime natural numbers which divide x, does not verify the Lipschitz condition.*

Proof. Let $K > 0$ be a given real number, $x > 2$ be a natural number which has the prime factorization

$$x = p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_r}^{\alpha_r}$$

and $y = x \cdot p_k$ where $p_k > \max\{2, K\}$ is a prime natural number which does not divide x.

We have:

$$|\theta(x) - \theta(y)| = \left|\theta\left(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_r}^{\alpha_r}\right) - \theta\left(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_r}^{\alpha_r} \cdot p_k\right)\right| =$$

$$\left|S(p_{i_1}^x) + S(p_{i_2}^x) + \cdots + S(p_{i_r}^x) - S(p_{i_1}^y) - S(p_{i_2}^y) - \cdots - S(p_{i_r}^y) - S(p_k^y)\right|.$$

But $x < x \cdot p_k = y$ which implies that $S(p_{i_j}^x) \leq S(p_{i_j}^y)$, for $j = \overline{1, r}$ so that

$$|\theta(x) - \theta(y)| = \left[S(p_{i_1}^y) - S(p_{i_1}^x)\right] + \left[S(p_{i_2}^y) - S(p_{i_2}^x)\right] + \cdots$$

$$+ \left[S(p_{i_r}^y) - S(p_{i_r}^x)\right] + S(p_k^y) =$$

$$= \left[S(p_{i_1}^{x \cdot p_k}) - S(p_{i_1}^x)\right] + \left[S(p_{i_2}^{x \cdot p_k}) - S(p_{i_2}^x)\right] + \cdots$$

$$+ \left[S(p_{i_r}^{x \cdot p_k}) - S(p_{i_r}^x)\right] + S(p_k^y).$$

In [1] it is proved the following formula which gives a lower and an upper bound for $S(p^r)$, wher p is a prime natural number and r is a natural number:

$$(p-1)r+1 \leq S(p^r) \leq pr \tag{1}$$

Using this formula, we have:

$$\begin{aligned} S(p_{i_j}^{x\cdot p_k}) - S(p_{i_j}^{x}) &\geq (p_{i_j}-1)\cdot x\cdot p_k + 1 - p_{i_j}\cdot x = \\ &= x(p_k(p_{i_j}-1) - p_{i_j}) > 0,\ (\forall)j = \overline{1,r} \end{aligned}$$

because $p_k > 2 \geq \frac{p_{i_j}}{p_{i_j}-1}$, $(\forall)j = \overline{1,r}$.

Then, we have:

$$|\theta(x)-\theta(y)| \geq S(p_k^y) \geq (p_k-1)\cdot x\cdot p_k >> (p_k-1)\cdot x\cdot K = K(p_k\cdot x - x) = K|x-y|$$

Therefore we have proved that for every real number $K > 0$ there exist the natural numbers x, y such that: $|\theta(x)-\theta(y)| > K|x-y|$ which shows that the function θ does not verify the Lipschitz condition.

Proposition 3 *The function* $\tilde{\theta} : N^* \to N$, $\tilde{\theta}(x) = \sum_{p_i \nmid x} S(p_i^x)$, *where* p_i *are the prime natural numbers which are smaller than* x *and do not divide* x*, does not verify the Lipschitz condition.*

Proof. Let $K > 0$ be a given real number. Then for $x > \frac{K}{2}$ and $y = 2\cdot x$, using the Tchebycheff theorem we know that between x and y there exists a prime natural number p. It is clear that p does not dividex and $2x$, thus $\tilde{\theta}(y)$ contains, in the sum, besides all the terms of $\tilde{\theta}(x)$, also $S(p^y)$ as a term. We have:

$$\left|\tilde{\theta}(x)-\tilde{\theta}(y)\right| = \left|\tilde{\theta}(x)-\tilde{\theta}(2x)\right| = \tilde{\theta}(2x)-\tilde{\theta}(x) \geq \tilde{\theta}(x) + S(p^y) - \tilde{\theta}(x) = S(p^y) \geq$$

$$\geq (p-1)y+1 = (p-1)\cdot 2x+1 \geq x\cdot 2x+1 = 2x^2+1 > x\cdot K = K|x-y|$$

therefore the function $\tilde{\theta}$ also does not verify the Lipschitz condition.

References

[1] Pål Gronås *A proof of the non-existence of SAMMA* , Smarandache Function Journal, vol.4-5, No.1, September 1994, pg.22-23.

[2] C Popovici *Teoria numerelor* , Editura didactică şi pedagogică, Bucureşti, 1973.

[3] F. Smarandache *An infinity of unsolved problems concerning a function in the number theory* , Smarandache Function Journal, vol.1, No.1, December 1994, pg.18-66.

[4] T. Yau *A problem of maximum* , Smarandache Function Journal, vol.4-5, No.1, September 1994, pg.45.

PROPERTIES OF THE NUMERICAL FUNCTION F_S

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In this paper are studied some properties of the numerical function $F_S(x): \mathbf{N} - \{0,1\} \to \mathbf{N}$ $F_S(x) = \sum\limits_{\substack{0<p\leq x \\ p \text{ prime}}} S_p(x)$, where $S_p(x) = S(p^x)$ is the Smarandache function defined in [4].

Numerical example: $F_S(5) = S(2^5) + S(3^5) + S(5^5)$; $F_S(6) = S(2^6) + S(3^6) + S(5^6)$.

It is known that: $(p-1)r + 1 \leq S(p^r) \leq pr$ so $(p-1)r < S(p^r) \leq pr$.

Than

$$x(p_1 + p_2 + \cdots + p_{\pi(x)} - \pi(x)) < F_S(x) \leq x(p_1 + p_2 + \cdots + p_{\pi(x)}) \tag{1}$$

Where $\pi(x)$ is the number of prime numbers smaller or equal with x.

PROPOSITION 1: The sequence $T(x) = 1 \quad \log F_S(x) + \sum\limits_{i=2}^{x} \frac{1}{F_S(i)}$ has limit $-\infty$.

Proof. The inequality $F_S(x) > x(p_2 + \cdots + p_{\pi(x)} - \pi(x))$ implies $-\log F_S(x) <$
$< -\log x(p_1 + p_2 + \cdots + p_{\pi(x)} - \pi(x)) < -\log x(\pi(x)p_1 - \pi(x)) = -\log x - \log \pi(x) - \log(p_1 - 1)$.

Than for $x=i$ the inequality (1) become:

$i(p_1 + \cdots + p_{\pi(i)} - \pi(i)) < F_S(i) \leq i(p_1 + \cdots + p_{\pi(i)})$ so:

$$\frac{1}{F_S(i)} < \frac{1}{i(p_1 + \cdots + p_{\pi(i)} - \pi(i))} < \frac{1}{i(p_1\pi(i) - \pi(i))} = \frac{1}{i\pi(i)(p_1 - 1)}$$

Than $T(x) < 1 - \log(x) - \log \pi(x) - \log(p_1 - 1) + \sum\limits_{i=2}^{x} \frac{1}{i\pi(i)(p_1 - 1)}$

$p_1 = 2 \Rightarrow T(x) = 1 - \log x - \log \pi(x) + \sum\limits_{i=2}^{x} \frac{1}{i\pi(i)}$

$\Rightarrow \lim\limits_{x\to\infty} T(x) \leq 1 - \lim\limits_{x\to\infty} \log x - \lim\limits_{x\to\infty} \log \pi(x) + \lim\limits_{x\to\infty} \sum\limits_{i=2}^{x} \frac{1}{i\pi(i)} = 1 - \infty - \infty + L = -\infty$.

PROPOSITION 2. The equation $F_S(x) = F_S(x+1)$ has no solution for $x \in \mathbf{N} - \{0,1\}$.

Proof. First we consider that $x+1$ is a prime number with $x > 2$. In the particular case $x = 2$ we obtain $F_S(2) = S(2^2) = 4$; $F_S(3) = S(2^3) + S(3^3) = 4 + 9 = 13$. So $F_2(2) < F_S(3)$.

Next we shall write the inequalities:

$$x(p_1+\cdots+p_{\pi(x)} - \pi(x)) \;<\; F_S(x) \;\le\; x(p_1+\cdots+p_{\pi(x)}) \qquad (2)$$

$$(x+1)(p_1+\cdots+p_{\pi(x)} + p_{\pi(x+1)} - \pi(x+1)) \;<\; F_S(x+1) \;\le\; (x+1)(p_1+\cdots+p_{\pi(x)} + p_{\pi(x+1)})$$

Using the reductio ad absurdum method we suppose that the equation $F_S(x) = F_S(x+1)$ has solution. From (2) results the inequalities

$$(x+1)(p_1+\cdots+p_{\pi(x)} + p_{\pi(x+1)} - \pi(x+1)) \;<\; F_S(x+1) \;\le\; x(p_1+\cdots+p_{\pi(x)}) \qquad (3)$$

From (3) results that:

$$x(p_1+\cdots+p_{\pi(x)}) - (x+1)(p_1+\cdots+p_{\pi(x)} + p_{\pi(x+1)} - \pi(x+1)) > 0$$

$$x(p_1+\cdots+p_{\pi(x)}) - x(p_1+\cdots+p_{\pi(x)}) - xp_{\pi(x+1)} + x\pi(x+1) - p_1 - \cdots - p_{\pi(x)} - p_{\pi(x+1)} +$$

$$+\pi(x+1) > 0.$$

But $p_{\pi(x+1)} > \pi(x+1)$ so the diference from above is negative for $x > 0$, and we obtained a contradiction. So $F_S(x) = F_S(x+1)$ has no solution for $x+1$ a prime number.

Next, we demonstrate that the equation $F_S(x) = F_S(x+1)$ has no solution for x and $x+1$ both composite numbers.

Let p be a prime number satisfing conditions $p > \frac{x}{2}$ and $p \le x-1$. Such p exists according to Bertrand's postulate for every $x \in \mathbf{N} - \{0,1\}$. Than in the factorial of the number $p(x-1)$, the number p appears at least x times.

So, we have $S(p^x) \le p(x-1)$.

But $p(x-1) < px + p - x$ (if $p > \frac{x}{2}$) and $px + p - x = (p-1)(x+1) + 1 \le S(p^{x+1})$. Therefore $\exists\, p \le x-1$ so that $S(p^x) < S(p^{x+1})$.

Than $F_S(x) = S(p_1^x)+\cdots+S(p^x)+\cdots+S(p_{\pi(x)}^x)$

$F_S(x+1) = S(p_1^{x+1})+\cdots+S(p^{x+1})+\cdots+S(p_{\pi(x)}^{x+1}) > F_S(x)$

In conclusion $F_S(x+1) > F_S(x)$ for x and $x+1$ composite numbers. If x is a prime number $\pi(x) = \pi(x+1)$ and the fact that the equation $F_S(x) = F_S(x+1)$ has no solution has the same demonstration as above.

Finally the equation $F_S(x) = F_S(x+1)$ has no solution for any $x \in \mathbf{N} - \{0,1\}$.

***PROPOSITION** 3.* The function $F_S(x)$ is strictly increasing function on its domain of definition.

The proof. of this property is justified by the proposition 2.

***PROPOSITION** 4.* $F_S(x+y) \;>\; F_S(x) + F_S(y) \quad \forall x,y \in \mathbf{N} - \{0,1\}$.

Proof. Let $x,y \in \mathbf{N} - \{0,1\}$ and we suppose $x < y$. According to the definition of $F_S(x)$ we have:

$$F(x+y) = S(p_1^{x+y}) + \cdots + S(p_{\pi(x)}^{x+y}) + S(p_{\pi(x)+1}^{x+y}) + \cdots + S(p_{\pi(y)}^{x+y}) + \tag{4}$$

$$+ S(p_{\pi(y)+1}^{x+y}) + \cdots + S(p_{\pi(x+y)}^{x+y})$$

$$F(x) + F(y) = S(p_1^{x}) + \cdots + S(p_{\pi(x)+1}^{x}) + S(p_1^{y}) + \cdots + S(p_{\pi(x)}^{y}) + S(p_{\pi(x)+1}^{y}) + \cdots + S(p_{\pi(y)}^{x})$$

But from (1) we have the following inequalities:

$$A = (x+y)(p_1 + \cdots + p_{\pi(x)} + p_{\pi(x)+1} + \cdots + p_{\pi(x+y)} - \pi(x+y)) < F(x+y) \leq$$

$$\leq (x+y)(p_1 + \cdots + p_{\pi(x)} + p_{\pi(x)+1} + \cdots + p_{\pi(x+y)}) \tag{5}$$

and

$$x(p_1 + \cdots + p_{\pi(x)} - \pi(x)) + y(p_1 + \cdots + p_{\pi(x)} + \cdots + p_{\pi(x)} + \cdots + p_{\pi(y)} - \pi(y)) \ < \ F(x) + F(y) \leq$$

$$\leq \ x(p_1 + \cdots + p_{\pi(x)}) + y(p_1 + \cdots + p_{\pi(x)} + p_{\pi(x)+1} + \cdots + p_{\pi(y)}) = B \tag{6}$$

We proof that B < A.

$$B < A \ \Leftrightarrow \ x(p_1 + \cdots + p_{\pi(x)}) + y(p_1 + \cdots + p_{\pi(x)}) + y(p_{\pi(x)+1} + \cdots + p_{\pi(y)})) <$$

$$x(p_1 + \cdots + p_{\pi(x)}) + y(p_1 + \cdots - p_{\pi(x)}) + x(p_{\pi(x)+1} + \cdots + p_{\pi(x+y)}) - x\pi(x+y) +$$

$$+ y(p_{\pi(x)+1} + \cdots + p_{\pi(y)}) + y(\, p_{\pi(y)+1} + \cdots + p_{\pi(x+y)}) - y\pi(x+y) \quad \Leftrightarrow$$

$$x(p_{\pi(x)+1} + \cdots + p_{\pi(x+y)} - \pi(x+y)) + y(p_{\pi(y)+1} + \cdots + p_{\pi(x+y)} - \pi(x+y)) \ > \ 0$$

But $p_{\pi(x+y)} \geq \pi(x+y)$ so that the inequality from above is true.

CONSEQUENCE: $F_S(xy) > F_S(x) + F_S(y) \qquad \forall\, x, y \in \mathbf{N} - \{0,1\}$
Because x and $y \in \mathbf{N} - \{0,1\}$ and $xy > x + y$ than $F_S(xy) > F_S(x+y) > F_S(x) + F_S(y)$

PROPOSITION 5. We try to find $\lim\limits_{n\to\infty} \dfrac{F_S(n)}{n^{\alpha}}$

We have $F_S(n) \ = \sum\limits_{\substack{0 < p_i \leq n \\ p_i = prime}} S(p_i^n)$ and:

$$\frac{p_1 + p_2 + \cdots + p_{\pi(n)} - \pi(n)}{n^{\alpha-1}} \ < \ \frac{F_S(n)}{n^{\alpha}} \ \leq \ \frac{p_1 + p_2 + \cdots + p_{\pi(n)}}{n^{\alpha-1}}$$

If $\alpha < 1$ than

$$\lim_{n\to\infty} n^{1-\alpha}(p_1 + \cdots + p_{\pi(n)} - \pi(n)) = \infty \cdot \infty = +\infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{F_S(n)}{n^{\alpha-1}} = +\infty.$$

If $\alpha = 1$ than

$$\lim_{n\to\infty} n^{1-\alpha}(p_1 + \cdots + p_{\pi(n)} - \pi(n)) = \lim_{n\to\infty} \ (p_1 + \cdots + p_{\pi(n)} - \pi(n)) = +\infty \Rightarrow \lim_{n\to\infty} \frac{F_S(n)}{n^{\alpha-1}} = +\infty$$

We consider now $\alpha > 1$.

We try to find $\lim_{n\to\infty} \frac{\sum_{i=1}^{\pi(n)} p_i - \pi(n)}{n^{\alpha-1}}$ and $\lim_{n\to\infty} \frac{\sum_{i=1}^{\pi(n)} p_i}{n^{\alpha-1}}$ appling Stolz - Cesaro:

Let $a_n = \sum_{i=1}^{\pi(n)} p_i - \pi(n)$ and $b_n = n^{\alpha-1}$.

$$\text{Than: } \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\sum_{i=1}^{\pi(n+1)} p_i - \pi(n+1) - \sum_{i=1}^{\pi(n)} p_i + \pi(n)}{(n+1)^{\alpha-1} - n^{\alpha-1}} = \begin{cases} \frac{n}{(n+1)^{\alpha-1} - n^{\alpha-1}} \\ \text{if } (n+1) \text{ is a prime} \\ 0, \text{ otherwise} \end{cases}.$$

Let $c_n = \sum_{i=1}^{\pi(n)} p_i$ and $d_n = n^{\alpha-1}$.

$$\text{Than } \frac{c_{n+1}-c_n}{d_{n+1}-d_n} = \frac{\sum_{i=1}^{\pi(n+1)} p_i - \sum_{i=1}^{\pi(n)} p_i}{(n+1)^{\alpha-1} - n^{\alpha-1}} = \frac{p_{\pi(n+1)}}{(n+1)^{\alpha-1} - n^{\alpha-1}} = \begin{cases} \frac{n+1}{(n+1)^{\alpha-1} - n^{\alpha-1}} \text{ if} \\ (n+1) \text{ is a prime} \\ 0, \text{ otherwise} \end{cases}.$$

First we consider the limit of the function.

$$\lim_{x\to\infty} \frac{x}{(x+1)^{\alpha-1} - x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{(\alpha-1)[(x+1)^{\alpha-2} - x^{\alpha-2}]} = 0 \quad \text{for } \alpha - 2 > 1$$

We used the l'Hospital theorem:
In the same way we have

$$\lim_{x\to\infty} \frac{x+1}{(x+1)^{\alpha-1} - x^{\alpha-1}} = 0 \quad \text{for } \alpha > 3.$$

So, for $\alpha > 3$ we have:

$$\lim_{x\to\infty} \frac{p_1 + p_2 + \cdots + p_{\pi(n)} - \pi(n)}{n^{\alpha-1}} = 0 \text{ and}$$

$$\lim_{x\to\infty} \frac{p_1 + p_2 + \cdots + p_{\pi(n)}}{n^{\alpha-1}} = 0. \quad \text{So } \lim_{x\to\infty} \frac{F(n)}{n^{\alpha}} = 0.$$

$$\text{Finally } \lim_{x\to\infty} \frac{F(n)}{n^{\alpha}} = \begin{cases} 0 & \text{for } \alpha > 3 \\ +\infty & \text{for } \alpha \le 1 \end{cases}$$

BIBLIOGRAPHY

[1] M. Andrei, C. Dumitrescu, V. Seleacu, L. Tuţescu, Şt. Zanfir — *Some remarks on the Smarandache Function*, Smarandache Function Journal, Vol. 4, No.1 (1994) 1-5;

[2] P. Gronas — *A note on* $S(p^r)$, Smarandache Function Journal, V. 2-3, No.1 (1993) 33;

[3] M. Andrei, I. Bălăcenoiu, C. Dumitrescu, E. Rădescu, N. Rădescu, V. Seleacu — *A linear combination with Smarandache Function to obtain the Identity*, Proceedings of 26th Annual Iranian Mathematic Conference Shahid Baharar University of Kerman, Kerman - Iran March 28 - 31 1995

[4] F. Smarandache — *A Function in the Number Theory*, An. Univ. Timişoara, Ser.St.Mat. Vol.XVIII, fasc. 1(1980) 9, 79-88.

ON A LIMIT OF A SEQUENCE OF THE NUMERICAL FUNCTION

by **Vasile Seleacu, Narcisa Vîrlan**

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In this paper is studied the limit of the following sequence:

$$T(n) = 1 - \log \sigma_S(n) + \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{1}{\sigma_S(p_i^K)}$$

We shall demonstrate that $\lim_{n\to\infty} T(n) = -\infty$.

We shal consider define the sequence $p_1 = 2, p_2 = 3, \dots, p_n =$ the nth prime number and the function $\sigma_S : \mathbf{N}^* \to \mathbf{N}$, $\sigma_S(x) = \sum_{\substack{d|x \\ d>0}} S(d)$, where S is Smarandache Function.

For example: $\sigma_S(18) = S(1) + S(2) + S(3) + S(6) + S(9) + S(18) =$ 0+2+3+3+6+6=20

We consider the natural number p_m^n, where p_m is a prime number. It is known that $(p-1)r + 1 \le S(p^r) \le pr$ so $S(p^r) > (p-1)r$.

Next, we can write $\sigma_S(p^r) = \sum_{s=0}^{r} S(p^s) > \sum_{s=0}^{r}(p-1)s = (p-1)\frac{r(r+1)}{2}$

$\sigma_S(p_i^k) > (p_i - 1)\frac{k(k+1)}{2}$, $\forall i \in \{1,\dots,m\}$, $\forall k \in \{1,\dots,n\}$.

$$\frac{1}{\sigma_S(p_i^k)} < \frac{2}{(p_i-1)k(k+1)}$$

This involves that:

$$\sum_{i=1}^{m}\sum_{k=1}^{n} \frac{1}{\sigma_S(p_i^k)} < \sum_{i=1}^{m}\sum_{k=1}^{n} \frac{2}{(p_i-1)k(k+1)} = \left(\sum_{i=1}^{m} \frac{1}{p_i-1}\right)\cdot\left(\sum_{k=1}^{n} \frac{2}{k(k-1)}\right)$$

$\sigma_S(k) > 0$, $\forall k \ge 2$ and $p_a^b \le p_m^n$ if $a \le m$ and $b \le n$ and $p_a^b = p_c^d$ if $a = c$ and $b = d$.

But $\sigma_S(p_m^n) > (p_m - 1)\frac{n(n+1)}{2}$ implies that $-\log \sigma_S(p_m^n) < -\log(p_m - 1)\frac{n(n+1)}{2}$ because $\log x$ is strictly increasing from 2 to $+\infty$.

Next, using inequality (1) we obtain

$$T(p_m^n) = 1 - \log \sigma_S(p_m^n) + \sum_{i=1}^{m}\sum_{k=1}^{n} \frac{1}{\sigma_S(p_i^k)} < 1 - \log(p_m - 1)\frac{n(n+1)}{2} +$$

$$+\left(\sum_{k=1}^{m}\frac{1}{p_k-1}\right)\cdot\left(\sum_{k=1}^{p_m}\frac{2}{k(k+1)}\right)$$

But $\sum_{k=1}^{p_m}\frac{2}{k(k+1)}=\frac{2p_m}{p_m+1}\quad\Rightarrow\quad T(p_m^{p_m})<1+\log 2-2\log p_m-\log(p_m-1)+$

$$+\frac{2p_m}{p_m+1}\sum_{k=1}^{m}\frac{1}{p_k-1}$$

$$T(p_m^{p_m})\ <\ 1+\log 2+2\left(-\log p_m+\sum_{k=1}^{p_m}\frac{1}{k}\right)+\frac{2p_m}{p_m+1}\cdot\sum_{k=1}^{m}\frac{1}{p_k-1}-2\sum_{k=1}^{p_m}\frac{1}{k}-\log(p_m-1)$$

We have $\sum_{k=1}^{m}\frac{1}{p_k-1}\ \le\ \sum_{k=1}^{p_m}\frac{1}{k}$.

So: $T(p_m^{p_m})<1+\log 2+2\left(-\log p_m+\sum_{k=1}^{p_m}\frac{1}{k}\right)+2\sum_{k=1}^{p_m}\frac{1}{k}\left(\frac{p_m}{p_m+1}-1\right)-\log(p_m-1)$

And then $\lim\limits_{m\to\infty}T(p_m^{p_m})\le 1+\log 2+2\lim\limits_{m\to\infty}(-\log p_m+\sum_{k=1}^{p_m}\frac{1}{k})-\lim\limits_{m\to\infty}\left[2\left(\sum_{k=1}^{p_m}\frac{1}{k}\right)\frac{1}{p_m+1}\right]-$

$-\lim\limits_{m\to\infty}\log(p_m-1)=1+\log 2+2\lim\limits_{p_m\to\infty}(-\log p_m+\sum_{k=1}^{p_m}\frac{1}{k})-\lim\limits_{p_m\to\infty}\left[\frac{2}{p_m+1}\left(\sum_{k=1}^{p_m}\frac{1}{k}\right)\right]-$

$-\lim\limits_{p_m\to\infty}\log(p_m-1)=1+\log 2+2\gamma-0-\infty=-\infty.$

It is known that $\lim\limits_{p_m\to\infty}\left(-\log p_m+\sum_{k=1}^{\infty}\frac{1}{k}\right)=\gamma$ (Euler's constant) and $\lim\limits_{p_m\to\infty}\left(\frac{2}{p_m+1}\cdot\sum_{k=1}^{p_m}\frac{1}{k}\right)=0$.

In conclusion $\lim\limits_{n\to\infty}T(n)=-\infty$.

REFERENCES

[1] F. Smarandache — *A Function in the Number Theory*, An. Univ. Timişoara, Ser.St.Mat. Vol.XVIII, fasc. 1(1980) 9, 79-88.

[2] — Smarandache Function Journal, Number Theory Publishing Co, Phoenix, New-York, Lyon, Vol.1, No.1, 1990.

[3] Pâl Gronas — *A proof the non-existence of "Samma"*, Smarandache Function Journal, V. 4-5, No.1 September (1994).

ON SOME SERIES INVOLVING SMARANDACHE FUNCTION

by

Emil Burton

The study of infinite series involving Smarandache function is one of the most interesting aspects of analysis.

In this brief article me give only a bare introduction to it.

First we prove that the series $\sum_{K=2}^{\infty} \frac{S(k)}{(kH)!}$ converges and has the sum $\sigma \in \left] e-\frac{5}{2}, \frac{1}{2} \right[$.

S(m) is the Smarandache function: $S(m) = \min\{k\in\mathbf{N}; m|k!\}$.

Let us denote $1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}$ by E_n. We show that

$E_{n+1}-\frac{5}{2} < \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < \frac{1}{2}$ as follows:

$$\sum_{k=2}^{n} \frac{k}{(k+1)!} = \sum_{k=2}^{n}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right) = \sum_{k=2}^{n} \frac{1}{k!}-\sum_{k=2}^{n} \frac{1}{(k+1)!} = \frac{1}{2!}-\frac{1}{(n+1)!}$$

$S(k) \le k$ implies that $\sum_{k=2}^{n} \frac{S(k)}{(k+1)!} \le \sum_{k=2}^{n} \frac{k}{(k+1)!} = \frac{1}{2}-\frac{1}{(k+1)!} < \frac{1}{2}$.

On the other hand $k \geq 2$ implies that $S(k) > 1$ and consequently:

$$\sum_{k=2}^{n} \frac{S(k)}{(k+1)!} > \sum_{k=2}^{n} \frac{1}{(k+1)!} = \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n+1}! = E_{n+1} - \frac{5}{2} .$$

It follows that $E_{n+1} - \frac{5}{2} < \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < \frac{1}{2}$ and therefore $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$ is a convergent series with sum $\sigma \in \left[e - \frac{5}{2}, \frac{1}{2} \right]$.

REMARK: Some of inequalities $S(k) \leq k$ are strictly and $k \geq S(k) + 1$, $S(k) \geq 2$. Hence $\sigma \in \left] e - \frac{5}{2}, \frac{1}{2} \right[$.

We can also check that $\sum_{k=r}^{\infty} \frac{S(k)}{(k-r)!}$, $r \in \mathbf{N}^*$ and $\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!}$, $r \in \mathbf{N}$, are both convergent as follows:

$$\sum_{k=r}^{n} \frac{S(k)}{(k-r)!} \leq \sum_{k=r}^{n} \frac{k}{(k-r)!} = \frac{r}{0!} + \frac{r+1}{1!} + \frac{r+2}{2!} + \ldots + \frac{r+(n-r)}{(n-r)!} =$$

$$= r\left(\frac{1}{0!} + \frac{1}{1!} + \ldots + \frac{1}{(n-r)!}\right) + \left(\frac{1}{1!} + \frac{2}{2!} + \ldots + \frac{n-r}{(n-r)!}\right) = rE_{n-r} + E_{n-r-1}$$

We get $\sum_{k=r}^{n} \frac{S(k)}{(k-r)!} < rE_{n-r} + E_{n-r-1}$ which that $\sum_{k=r}^{\infty} \frac{S(k)}{(k+ \quad)!}$ converges.

Also we have $\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!} < \infty$, $r \in \mathbf{N}$.

Let us define the set $M_2 = \left\{ m \in \mathbf{N} : m = \frac{n!}{2}, n \in \mathbf{N}, n \geq 3 \right\}$.

If $m \in M_2$ it is obvious that

$$S(m) = n, \quad m = \frac{n!}{2}.\ m \in M_2 \rightarrow \frac{m}{S(m)!} = \frac{\frac{n!}{2}}{n!} .$$

So, $\sum_{\substack{m=3 \\ m \in M_2}}^{\infty} \frac{m}{S(m)!} = \infty$ and therefore $\sum_{\substack{k=2 \\ k \in \mathbf{N}}}^{\infty} \frac{k}{S(k)!} = \infty$.

A problem: test the convergence behaviour of the series

$$\sum_{\substack{k=2 \\ k \in \mathbf{N}}}^{\infty} \frac{1}{S(k)!} .$$

REFERENCES

1. Smarandache Function Journal, Vol.1 1990, Vol. 2-3, 1993, Vol.4-5 1994, Number Theory Publishing, Co., R. Muller Editor, Phoenix, New York, Lyon.

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SOME PROPERTIES OF SMARANDACHE FUNCTIONS OF THE TYPE I

by

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We consider the construction of Smarandache functions of the type I S_p ($p \in \mathbb{N}^*$, p prim) which are defined in [1] and [2] as follows:

$$S_n : \mathbb{N}^* \longrightarrow \mathbb{N}^* \quad ; \quad S_1(k) = 1 \quad ; \quad S_n(k) = \max_{1 \le j \le r} \{ S_{p_j}(i_j . k) \}$$

$$\text{for} \qquad n = p_1^{i_1} p_2^{i_2} \ldots p_r^{i_r}$$

In this paper there are presented some properties of these functions. We shall study the monotonicity of each function S_n and also the monotonicity of some subsequences of the sequence $(S_n)_{n \in \mathbb{N}^*}$.

1. Proposition. The function S_n is monotonous increasing for every positiv integer n.

Proof. The function S_1 is abviously monotonous increasing.

Let $k_1 < k_2$ where $k_1, k_2 \in \mathbb{N}^*$. Supposing that n is a prime number and taking accont that $(S_n(k_2))! =$ multiple n^{k_1} = multiple n^{k_2},

it results that $S_n(k_1) \le S_n(k_2)$, therefore S_n is monotonous increasing. Let

$$S_n(k_1) = \max_{1\le j\le k} \{ S_{p_j}(i_j \cdot k_1) \} = S_{p_m}(i_m \cdot k_1)$$

$$S_n(k_2) = \max_{1\le j\le r} \{ S_{p_j}(i_j \cdot k_2) \} = S_{p_t}(i_t \cdot k_2)$$

Because $S_{p_m}(i_m \cdot k_1) \le S_{p_m}(i_m \cdot k_2) \le S_{p_t}(i_t \cdot k_2)$

it results that $S_n(k_1) \le S_n(k_2)$ so S_n is monotonous increasing.

2. Proposition. The sequence of functions $(S_{p^i})_{i\in\mathbb{N}^*}$ is monotonous increasing, for every prime number p.

Proof. For any two nombers $i_1, i_2 \in \mathbb{N}^*$, $i_1 < i_2$ and for any $n \in \mathbb{N}^*$ we have :

$S_{p^{i_1}}(n) = S_p(i_1 \cdot n) \le S_p(i_2 \cdot n) = S_{p^{i_2}}(n)$ therefore $S_{p^{i_1}} \le S_{p^{i_2}}$.

Hence the sequence $\{S_{p^i}\}_{i\in\mathbb{N}^*}$ is monotonous increasing for every prime number p.

3. Proposition. Let p and q two given prime numbers. If $p<q$ then

$$S_p(k) < S_q(k) \quad , \quad k \in \mathbb{N}^*$$

Proof. Let the sequence of coefficients (see [2]) $a_1^{(p)}, a_2^{(p)}, \ldots, a_s^{(p)}, \ldots$

Every $k \in \mathbb{N}^*$ can be uniquely written as

$$k = t_1 a_s^{(p)} + t_2 a_{s-1}^{(p)} + \ldots + t_s a_1^{(p)} \qquad (1)$$

where $0 \le t_i \le p-1$, for $i = \overline{1,s-1}$, and $0 \le t_s \le p$.

The procedure of passing from k to $k+1$ in formule (1) is :

(i) t_s is increasing with a unity.

(ii) if t_s can not increase with a unity, then t_{s-1} is increasing with a unity and $t_s = 0$

(iii) if neithe t_s, nor t_{s-1} are not increasing with a unity then t_{s-2} is increasing with a unity and $t_s = t_{s-1} = 0$

The procedure is continued in the same way until we obtain the expresion of $k+1$.

Denoting $\Delta_k(S_p) = S_p(k+1) - S_p(k)$ the leap of the function S_p when we pass from k to $k+1$ corresponding to the procedure described above. We find that

- in the case (i) $\Delta_k(S_p) = p$
- in the case (ii) $\Delta_k(S_p) = 0$
- in the case (iii) $\Delta_k(S_p) = 0$

. . . .

It is abviously seen that: $S_p(n) = \sum_{k=1}^{n} \Delta_k(n_p) + S_p(1)$.

Analogously we write $S_q(n) = \sum_{k=1}^{m} \Delta_k(n_q) + S_q(1)$

Taking into account that $S_p(1) = p < q = S_q(1)$ and using the procedure of passing from k to $k+1$ we deduce that the number of leaps with zero value of S_p is greater then the number of leaps with zero value of S_q, respectively the number of leaps with value p of S_p is less then the number of leaps of S_q with value

q it result that

$$\sum_{k=1}^{n} \Delta_k(S_p) + S_p(1) < \sum_{k=1}^{n} \Delta_k(S_q) + S_q(1) \qquad (2)$$

Hence $S_p(n) < S_q(n)$, $n \in \mathbb{N}^*$.

As an example we give a table with S_2 and S_3 for $0<n<21$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
the leap		2	0	2	2	0	0	2	2	0	2	2	0	0	0	2	0	2	2	2
$S_2(k)$	2	4	4	6	8	8	8	10	12	12	14	16	16	16	16	18	18	20	22	24
the leap		3	3	0	3	3	3	0	3	3	3	0	0	3	3	3	0	3	3	3
$S_3(k)$	3	6	9	9	12	15	18	18	21	24	27	27	27	30	33	36	36	39	42	45

Hence $S_2(k) < S_3(k)$ for $k = 1,2,\dots,20$.

4. Remark. For any monotonous increasing sequence of prime numbers $p_1 < p_2 < \dots < p_n < \dots$ it results that

$$S_1 < S_{p_1} < S_{p_2} < \dots < S_{p_n} < \dots$$

If $n = p_1^{i} \; p_2^{i} \dots p_t^{i}$ and $p_1 < p_2 < \dots < p_t$ then

$$S_n(k) = \max_{1 \le j \le t} \{ S_{p_j^i}(k) \} = S_{p_t^i}(k) = S_{p_t}(ik)$$

5. Proposition. If p and q are prime numbers and $p.i < q$ then $S_{p^i} < S_q$.

Proof. Because $p.i < q$ it results

$$S_{p^i}(1) \le p.i < q = S_q(1) \qquad (3)$$

and $S_{p^i}(k) = S_p(ik) \le i\, S_p(k)$.

From (3) passing from k to $k+1$, we deduce

$$\Delta_k(S_{p^i}) \le i\, \Delta_k(S_p) \qquad (4)$$

Taking into account the proposition 3. from (4) it results that when we pass from k to $k+1$ we obtain

$\Delta_k(S_{p^i}) \le i\,\Delta_k(S_p) \le i.p < q$ and $i\sum_{k=1}^{n} \Delta_k(S_p) \le \sum_{k=1}^{n} \Delta_k(S_q)$ (5)

Because we have

$$S_{p^i}(n) = S_{p^i}(1) + \sum_{k=1}^{n} \Delta_k(S_{p^i}) \le S_{p^i}(1) + i \sum_{k=1}^{n} \Delta_k(S_p)$$

and

$$S_q(n) = S_q(1) + \sum_{k=1}^{n} \Delta_k(S_q)$$

from (3) and (5) it results $S_{p^i}(n) \le S_q(n)$, $n \in \mathbb{N}^*$

6. Proposition. If p is a prime number then $S_n < S_p$ for every $n < p$.

Proof. If n is a prime number from $n < p$, using the proposition 3 it results $S_n(k) < S_p(k)$ for $k \in \mathbb{N}^*$. If n is a composed, that is $n = p_1^{i_1} \ldots p_t^{i_t}$ then $S_n(k) = \max_{1 \le j \le t} \{ S_{p_j^{i_j}}(k) \} = S_{p_r^{i_r}}(k)$.

Because $n < p$ it results $p_r^{i_r} < p$ and using the proposition 5 and knowing that $i_r p_r \le p_r^{i_r} < p$ it results that $S_{p_r^{i_r}}(k) \le S_p(k)$ therefore for $k \in \mathbb{N}^*$ $S_n(k) < S_p(k)$.

References

[1] Balacenoiu I , Smarandache Numerical Functions in Smarandache Function Journal nr. 4 / 1994.

[2] Smarandache F., A function in the number theory. " An. Univ. Timisoara" vol XVIII , fasc 1, pp 79-88.

SOME PROBLEMS ON SMARANDACHE FUNCTION

by

Charles Ashbacher

In this paper we shall investigate some aspects involving Smarandache function, $S:\mathbf{N}^* \dashrightarrow \mathbf{N}^*$, $S(n) = \min\{m \mid n \text{ divide } m!\}$.

1. THE MINIMUM OF S(n)/n

Which is minimumum of $S(n)/n$ if $n > 1$?

1.1. **THEOREM:**

a) $S(n)/n$ has no minimum for $n > 1$.

b) $\lim S(n)/n$ as n goes to infinity does not exist.

Proof:

a) Since $S(n) > 1$ for $n>1$ it follows that $S(n)/n > 0$. Assume that $S(n)/n$ has a minimum and let the rational fraction be represented by r/s. By the infinitude of the natural numbers, we can find a number m such $2/m < r/s$. Using the infinitude of the primes, we can find a prime number $p > m$. Therefore, we have the sequence

$$2/p < 2/m < r/s$$

We have $S(p \cdot p) = S(p^2) = 2p$. It is known that $S(p \cdot p)=2p$. The ratio of $S(p^2)/(p \cdot p)$ is then

$$2p/(p^2) = 2/p$$

And this ratio is less than r/s, contradicting the assumption of the minimum.

b) Suppose lim $S(n)/n$ exists and has value r. Now choose, $e > 0$ and $e < 1/p$ where p is a twenty digit prime. Since $S(p) = p$, $S(p)/p = 1$.
However, $S(p \cdot p) = 2p$, so the ratio $S(n)/n = 2p/(p \cdot p) = 2/p$. Since p is a twenty digit prime,

$$| S(p)/p - S(p \cdot p)/(p \cdot p) | > e \text{ by choice of } e .$$

so the limit does not exist.

2. THE DECIMAL NUMBER WHOSE DIGITS ARE THE VALUES OF SMARANDACHE FUNCTION IS IRRATIONAL.

Unsolved problem number (8) in [1] is as follows:

Is $r = 0,0234537465114...$, where the sequence of digits is $S(n)$, $n \geq 1$, an irrational number?

The number r is indeed irrational and this claim will be proven below.

The following well-known results will be used.

DIRICHLET'S THEOREM:

If $d > 1$ and $a \neq 0$ are integers that are relativey prime, then the arithmetic progression

$$a, a, + d, a + 2d, a + 3d, \ldots$$

contains infinitely many primes.

Proof of claim:

Assume that r as defined above is rational. Then after some m digits, there must exist a series of digits $t_1, t_2, t_3, \ldots, t_n$, such that

$$r = 0{,}023453746114\ldots s\overline{t_1 t_2 t_3 t_4 t_5 \ldots t_n}$$

where s is the m-th digit in the decimal expansion.

Now, construct the repunit number consisting of 10n 1's.

$$\mathbf{a = \underbrace{11111 \ldots 111}_{10n\ times}}$$

and let $\mathbf{d = \underbrace{1000 \ldots 00}_{10n + 1\ 0's}}$

Since the only prime factors of d are 2 and 5, it is clear that a and d are relatively prime and by Dirichlet's Theorem, the sequence

$$\mathbf{a, a + d, a + 2d, \ldots}$$

must contain primes. Given the number of 1's in a and the fact that S(p) = p, it follows that the sequence of repeated digits in r must consist entirely of 1's.

Now, construct the repdigit number constructed from 10n 3's

$$a = \underbrace{3333\ldots333}_{10n\ times}$$

and using

$$d = \underbrace{10000\ldots00}_{10n+1\ 0's}$$

we again have a and d relatyvely prime. Arguments similar to those used before forces the conclusion that the sequence of repeted digits must consist entirely of 3's.

This is of course impossibile and therefore the assumption of rationality must be false.

3. ON THE DISTRIBUTION OF THE POINTS OF S(n)/n IN THE INTERVAL (0,1).

The following problem is listed as unsolved problem number (7) in [1]

Are the points $p(n) = S(n)/n$ uniformly distributed in the interval (0,1)?

The answer is no, the interval (0.5,1.0) contains only a finite number of points p(n).

3.1. **LEMMA:**

$$\frac{S(p^k)}{p^k} \geq \frac{S(p^{k+1})}{p^{k+1}} .$$

For p prime and k>0.

Proof:

It is well-known that $S(p^k) = j \cdot p$ where $j \le k$.

Therefore, forming the expressions

$$\frac{S(p^k)}{p^k} = \frac{j \cdot p}{p^k} = \frac{j}{p^{k-1}}$$

$$\frac{S(p^{k+1})}{p^{k+1}} = \frac{m \cdot p}{p^{k+1}} = \frac{m}{p^k}$$

where m must have one of the two values $\{j, j+1\}$.

With the restrictions on the values of m and p, it is clear that

$$\frac{j}{m} \ge \frac{1}{p}$$

which implies that

$$\frac{S(p^k)}{p^k} \ge \frac{S(p^{k+1})}{p^{k+1}}$$

which is the desired result. Equality occurs only when p=2, j=1 and m=2.

3.2. **LEMMA:**

The interval (0.5,1.0) contains only a finite number of points p(n), where

$$p(n) = \frac{S(n)}{n}$$ and n is a power of a prime.

Proof:

If n=p $\frac{S(p)}{p} = 1$, outside the interval.

Start with the smallest prime p=2 and move up the powers of 2

$$\frac{S(2 \cdot 2)}{(2 \cdot 2)} = \frac{4}{4} = 1$$

$$\frac{S(2 \cdot 2 \cdot 2)}{(2 \cdot 2 \cdot 2)} = \frac{4}{8}$$

$$\frac{S(2\cdot2\cdot2\cdot2)}{(2\cdot2\cdot2\cdot2)} = \frac{6}{16} < 0.5 \ .$$

And applying the previous lemma, all additional powers of 2 will yield a value less than 0.5.

Taking the next smallest prime p=3 and moving up the powers of 3

$$\frac{S(3\cdot3)}{(3\cdot3)} = \frac{6}{9}$$

$$\frac{S(3\cdot3\cdot3)}{(3\cdot3\cdot3)} = \frac{9}{27} < 0.5$$

and by the previous lemma, all additional powers of 3 also yield a value less than 0.5.

Now, if p>3 and p is prime

$$\frac{S(p\cdot p)}{(p\cdot p)} = \frac{2}{p} < 0.5$$

so all other powers of primes yield values less than 0.5 and we are done.

3.3. **THEOREM:**

The interval (0.5,1.0) contains only a finite number of points p(n) where

$$p(n) = \frac{S(n)}{n} \ .$$

Proof:

It is well-known that if

$$n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot p_3^{\alpha_3}\cdot \ldots \cdot p_n^{\alpha_n} \ , \ \text{then}$$

$$S(n) = \max\{S(p_i^{\alpha_i})\} \ .$$

Applying the well-known result with the formula for p(n)

$$p(n) = \frac{S(p_i^{\alpha_i})}{p_i^{\alpha_i}} \cdot \frac{1}{\prod_{j\neq i} p_j^{\alpha_j}}$$

which is clearly less than

$$\frac{S(p_i^{\alpha_i})}{p_i^{\alpha_i}} .$$

Theorefore, applying Lemma 2, we get the desired results.

3.4. **COROLLARY:**

The points $p(n)=S(n)/n$ are not evenly distributed in the interval $(0,1)$.

4. THE SMARANDACHE FUNCTION DOES NOT SATYSFY A LIPSCHITZ CONDITION

Unsolved problem number 31 in [1] is as follows.

Does the Smarandache function veryfy a Lipschitz condition? In other words, is there a real number L such that

$$| S(m) - S(n) | \le L \, | m - n | \text{ for all } m,n \text{ in } \{0,1,2,3,\ldots\}.$$

4.1. **THEOREM**

The Smarandache function does not verify a Lipschitz condition.

Proof:

Suppose that Smarandache function does indeed satisfy a Lipschitz condition and let L be the Lipschitz constant.

Since the numbers of primes is infinite, is possible to fiind a prime p such that

$$p - (p + 1)/2 > L$$

Now, examine the numbers (p-1) and (p+1). Clearly, at least one must not be a power of two, so we choose that one call it m.

Factoring m into the product of all primes equal to 2 and everything else, we have

$$m = 2^k \cdot n$$

Then $S(m) = \max\{S(2^k), S(n)\}$ and because $S(2^k) \le 2^k$.
we have

$$S(m) \le \frac{m}{2}$$

And so,

$$|S(p) - S(m)| > |p - \frac{m}{2}| > L$$

Since $|p - m| = 1$ by choice of m, we have a violation of the Lipschitz condition, rendering our original assumption false.

Therefore, the Smarandache function does not satisfy a Lipschitz condition.

5. ON THE SOLVABILITY OF THE EXPRESSION S(m) =n!

One of the unsolved problems in [1] involves a relationship between the Smarandache and factorial functions.

Solve the Diophantine Equation

$$S(m) = n!$$

where m and n are positive integers.

This equation is always solvable and the number of solutions is a function of the number of primes less than or equal to n.

5.1. **LEMMA**: Let be a prime. Then the range of the sequence

$$S(p), S(p \cdot p), S(p \cdot p \cdot p), \ldots$$

will contain all positive integral multiples of p.

Proof: It has already been proven [2] that for all integers $k > 0$, there exists another integer $m > 0$, such that

$$S(p^k)k = mp \quad \text{where} \quad m \le k$$

and in particular

$$S(p) = p$$

So the only remaining element of the proof is to show that m takes on all possible integral values greater than 0.

Let p be an arbitrary prime number and define the set $M = \{$ all positive integers n such that there is no positive integer k such that $S(p^k) = mp$ $\}$ and assume that M is not empty.

Since M is non-empty subset of the natural numbers, it must have a least element. Call that least element m. It is clear that $m > 1$.

Now, let j be the largest integer such that

$$S(p^j) = (m-1) \cdot p$$

and consider the exponent $j + 1$.

By the choise of j, it follows that either

1) $S(p^{j+1}) = m \cdot p$

or

2) $S(p^{j+1}) = n \cdot p$ where $n > m$

in the first case, we have a contradiction of our choise of m,

so we proceed to case (2).

However, it is a direct consequence of the definition of prime numbers that if $((m - 1)\cdot p)!$ contains j instances of the prime p, then $m\cdot p$ is the smallest number such that $(m\cdot p)!$ contains more than j instances of p. Then, using the definition of Smarandache function where we choose the smallest number having the required number of instances we have a contradiction of case (2).

Therefore, it follows that there can be no least element of the set M, so M must be empty.

5.2.**THEOREM:** Let n be any integer and p a prime less than or equal to n. Then, there is some integer k such that

$$S(p^k) = n!$$

Therefore, each equation of the form $S(m) = n!$ has at least $^{\#}p$ solutions, where $^{\#}p$ is the number of primes less than or equal to n.

Proof:

Since n! is an integral multiple of p for p any prime less than or equal to n, this is a direct consequence of the lemma.

Now that the question is known to have multiple solutions, the next logical question is to determine how many solutions there are.

5.3. **DEFINITION:** Let NSF(n) be the number of integers m, such that $S(m) = n!$.

From the fact hat $S(n) = \max\{S(p_i^{a_i})\}$, we have the following obvious result.

Corollary:

Let n be a positive integer, q a prime less than or equal to n and k another positive integer such that $S(q^k) = n!$. Then, all numbers having the prime factorization form $m = q^k p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r}$ where $S(q^k) > S(p_i^{\alpha_i})$ will also be solutions the equation S(m) = n!

To proceed further, we need the following two obvious lemmas.

5.4. **LEMMA:** If p is a prime and m and n nonnegative integers m > n, then $S(p^n) \leq S(p^m)$.

5.5. **LEMA:** If p and q are primes such that p < q and k > 0, then $S(p^k) < S(q^k)$.

The following theorem gives an initial indication regarding how fast NSF(n) grows as n does.

5.6. **THEOREM:** Let q be a prime number and k an exponent such that $S(q^k) = n!$ Let $p_1, p_2, \ldots, p_r$ be the list of primes less than q. Then the number of solutions to the equation S(m) = n! where m contains exactly k instances of the prime q is at least $(k+1)^r$.

Proof: Applyng the two lemmas, the numbers $m = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r} q^k$ where all of exponents on the primes p_i are at most solutions to the equation. Since each prime pi can have (k + 1), {0,1,2,...,k} different values for the exponent, simple counting gives the result.

Since this procedure can be repeated for each prime less than or equal to n, we have an initial number of solutions given by the formula

$$\sum_{i=2}^{s} (k_i+1)^{i-1} + 1$$

where s is the number of primes less then or equal to n, k is the integer such that

$$S(p_j^{K_i}) = n!$$

And even this is a very poor lower bound on the number of solutions for n having any size.

5.7. **COROLARY:** Let q be a prime such that for some k $S(q^k) = n!$. Then if p is any prime such that there is some integer j such that $S(p^j) < S(q^k)$, then the product of any solution and p any power less than or equal to j will also be a solution.

Proof: Clear.

If q is the largest prime less than or equal to n, it is easy to show for "large" n that there are primes p > n > q that satisfy the above conditions. If p is any prime, then by Bertrand's Postulate, another prime r can be found in the interval p > r > 2p. Since q < n < 2n < n! for n > 2 and S(p) = p, we have one such prime. Expanding this reasoning, it folows that the number of such primes is at least j, where j is the largest exponent of 2 such that $q2^j \le n!$, or put another way, the largest power of 2 that is less than or equal to n!/q.

Since there are so many solutions to the equation S(m) = n!, it is logical to consider the order of growth of the number of solutions rather than the actual number.

It is well known that the number of primes less than or equal to n is asymptotic to the ratio n/ln(n). Now, let p be the largest prime less than n. As n gets larger, it is clear that the factor m

such that mp = n! grows on the order of a factorial. Since $m \leq k$, where k is the exponent on the power p, it follows that the number grows on the order of the product of factorials. Since the number of items in the product depends on the number of primes q such that $q < mp = n!$, it follows that this number also grows on the order of a factorial.

Putting it all together, we have the following behavior of NSF(n).

NSF(n) grows on the order of product of items all on the order of the factorial of n, where the number of elements in the product also grows on the order of a factorial of n.

Cleary, this function grows at an astronomical rate.

6. THE NUMBER OF PRIMES BETWEEN S(n) and S(n+1)

I read the letter by I.M.Radu that appeared in [3] stating that there is always a prime between S(n) and S(n+1) for all numbers 0<n<4801, where S(n) is the Smarandache function.

Since 1 have a computer program that computes the values of S(n), I decided to investigate the problem further. The serch was conducted up through n<1,033,197 and for instances where there is no prime p, where $S(n) \leq p \leq S(n+1)$. They are as follows:

$n=224=2\cdot2\cdot2\cdot2\cdot2\cdot7$ $S(n)=8$ $n=225=3\cdot3\cdot5\cdot5$ $S(n)=10$

$n=2057=11\cdot11\cdot17$ $S(n)=22$ $n=2058=2\cdot3\cdot7\cdot7\cdot7$ $S(n)=21$

$n=265225=5\cdot5\cdot103\cdot103$ $S(n)=206$ $n=265226=2\cdot13\cdot101\cdot101$ $S(n)=202$

$n=843637=37\cdot151\cdot151$ $S(n)=302$

$n=843638=2\cdot19\cdot149\cdot149$ $S(n)=298$

As can be seen, the first two values contradict the assertion made by I.M.Radu in his letter. Notice that the last two cases involve pairs of twin primes. This may provide a clue in the search for additional solutions.

7. ADDITIONAL VALUES WHERE THE SMARANDACHE FUNCTION SATISFIES THE FIBONACCI RELATIONSHIP S(n)+S(n+1)=S(n+2)

In [4] T.Yau poses the following problem:

For what triplets n, n+1 and n+2 does the Smarandache function satisfy the Fibonacci relationship

$$S(n)+S(n+1) = S(n+2) \ ?$$

Two solutions

$$S(9)+S(10) = S(11) \quad 6+5 = 11$$

$$S(119)+S(120) = S(121) \quad 17+5 = 22$$

were found, but no general solution was given.

To further investigate this problem, a computer program was written that tested all values for n up to 1,000,000. Additional solutions were found and all known solutions with their prime factorizations appear in the table below.

$S(9)+S(10) = S(11)$ $9 = 3\cdot3$ $10 = 2\cdot5$ $11 = 11$

$S(119) + S(120) = S(121)$ $119 = 7\cdot17$ $120 = 2\cdot2\cdot2\cdot3\cdot5$ $121 = 11\cdot11$

S(4900) + S(4901) = S(4902); S(26243) + S(26244) = S(26245)

S(32110) + S(32111) = S(32112); S(64008) + S(64009) = S(64010);

S(368138) + S(368139) = S(368139); S(415662) + S(415663) = S(415664);

I am unable to discern a pattern in these numbers that would lead to a proof that there is an infinite family of solutions. Perhaps another reader will be able to do so.

8. WILL SOME PROBLEMS INVOLING THE SMARANDACHE FUNCTION ALWAYS REMAIN UNSOLVED?

The most unsolved problems of the same subject are related to the **Smarandache function** in the Analytic Number Theory:

$S:Z$---N , S(n) is defined as the smallest integer such that S(n)! is divisible by n.

The number of these unsolved problems concerning the function is equal to... an infinity!! Therefore, they will never be all solved!

One must be very careful in using such arguments when dealing with infinity. As is the case with number theoretic functions, a result in one area can have many aplications to other problems. The most celebrated recent instance is the "prof" of "Fermat's Last Theorem". In this case a result in elliptical functions has the proof as a consequence.

Since S(n) is still largely unexplored, it is quite possible

that the resolution of one problem leads to the resolution of many, perhaps infinitely many, others. If that is indeed the case, then all problems may eventually be resolved.

REFERENCES

1. R. Muller : Unsolved Problems Related to Smarandache Function (Number Theory Publishing Co., Glendale, Az, USA, 1993).
2. P Gronas : A note on $S(p^r)$ (Smarandache Function J., V. 2-3, Nr.1, (1993), 33).
3. M. Radu : Letter to the Editor (Math Spectrum, V.27, Nr.2, (1994/95), 44).
4. T. Yau : A problem Concerning the Fibonacci Series (Smarandache Function J., V. 4-5, Nr.1, (1994), 42).

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ABOUT THE SMARANDACHE SQUARE'S COMPLEMENTARY FUNCTION

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DEFINITION 1. *Let $a:\mathbf{N}^* \to \mathbf{N}^*$ be a numerical function defined by $a(n) = k$ where k is the smallest natural number such that nk is a perfect square: $nk = s^2$, $s \in \mathbf{N}^*$, which is called the Smarandache square's complementary function.*

PROPERTY 1. *For every $n \in \mathbf{N}^*$ $a(n^2) = 1$ and for every prime natural number $a(p)=p$.*

PROPERTY 2. *Let n be a composite natural number and $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}}$, $0 < p_{i_1} < p_{i_2} < \cdots < p_{i_r}$, $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r} \in \mathbf{N}$ it's prime factorization. Then*

$$a(n) = p_{i_1}^{\beta_{i_1}} \cdot p_{i_2}^{\beta_{i_2}} \cdots p_{i_r}^{\beta_{i_r}} \text{ where } \beta_{i_j} = \begin{cases} 1 \text{ if } \alpha_{i_j} \text{ is an odd natural number} \\ \\ 0 \text{ if } \alpha_{i_j} \text{ is an even natural number} \end{cases} \quad j = \overline{1,r}.$$

If we take into account of the above definition of the function a, it is easy to prove both the properties.

PROPERTY 3. $\dfrac{1}{n} \leq \dfrac{a(n)}{n} \leq 1$, for every $n \in \mathbf{N}^*$ *where a is the above defined function.*

Proof. It is easy to see that $1 \leq a(n) \leq n$ for every $n \in \mathbf{N}^*$, so the property holds.

CONSEQUENCE. $\sum\limits_{n \geq 1} \dfrac{a(n)}{n}$ diverges.

PROPERTY 4. *The function $a:\mathbf{N}^* \to \mathbf{N}^*$ is multiplicative:*

$$a(x \cdot y) = a(x) \cdot a(y) \text{ for every } x, y \in \mathbf{N}^* \text{ whith } (x,y) = 1$$

Proof. For $x = 1 = y$ we have $(x,y) = 1$ and $a(1 \cdot 1) = a(1) \cdot a(1)$. Let $x = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}}$ and $y = q_{j_1}^{\gamma_{j_1}} \cdot q_{j_2}^{\gamma_{j_2}} \cdots q_{j_s}^{\gamma_{j_s}}$ be the prime factorization of x and y, respectively, and $x \cdot y \neq 1$. Because $(x,y) = 1$ we have $p_{i_h} \neq q_{j_k}$ for every $h = \overline{1,r}$ and $k = \overline{1,s}$. Then,

$$a(x)=p_{i_1}^{\beta_{i_1}}\cdot p_{i_2}^{\beta_{i_2}}\cdots p_{i_r}^{\beta_{i_r}} \text{ where } \beta_{i_j}=\begin{cases}1 \text{ if } \alpha_{i_j} \text{ is odd}\\ 0 \text{ if } \alpha_{i_j} \text{ is even}\end{cases},\ j=\overline{1,r},$$

$$a(y)=q_{j_1}^{\delta_{j_1}}\cdot q_{j_2}^{\delta_{j_2}}\cdots q_{j_s}^{\delta_{j_s}} \text{ where } \delta_{j_k}=\begin{cases}1 \text{ if } \gamma_{j_k} \text{ is odd}\\ 0 \text{ if } \gamma_{j_k} \text{ is even}\end{cases},\ k=\overline{1,s} \text{ and}$$

$$a(xy)=p_{i_1}^{\beta_{i_1}}\cdot p_{i_2}^{\beta_{i_2}}\cdots p_{i_r}^{\beta_{i_r}}\cdot q_{j_1}^{\delta_{j_1}}\cdot q_{j_2}^{\delta_{j_2}}\cdots q_{j_s}^{\delta_{j_s}}=a(x)\cdot a(y)$$

Property 5. *If $(x,y)=1$, x and y are not perfect squares and x,y>1 the equation a(x)=a(y) has not natural solutions.*

Proof. It is easy to see that $x\neq y$. Let $x=\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}$ and $y=\prod_{k=1}^{s}q_{j_k}^{\gamma_{j_k}}$, (where $p_{i_h}\neq q_{j_k}$, $\forall h=\overline{1,r}, k=\overline{1,s}$ be their prime factorization.

Then $a(x)=\prod_{h=1}^{r}p_{i_h}^{\beta_{i_h}}$ and $a(y)=\prod_{k=1}^{s}q^{\delta_{j_k}}$, where β_{i_h} for $h=\overline{1,r}$ and δ_{j_k} for $k=\overline{1,s}$ have the above signifiecance, but there exist at least $\beta_{i_h}\neq 0$ and $\delta_{j_k}\neq 0$. (because x and y are not perfect squares). Then $a(x)\neq a(y)$.

Remark. If x=1 from the above equation it results $a(y)=1$, so y must be a a perfect square (analogously for y=1).

Consequence. The equation $a(x)=a(x+1)$ has not natural solutions, because for x>1 x and x+1 are not both perfect squares and (x, x+1)=1.

Property 6. *We have $a(x\cdot y^2)=a(x)$, for every $x,y\in\mathbf{N}^*$.*

Proof. If (x, y)=1, then $(x,y^2)=1$ and using property 4 and property 1 we have $a(x\cdot y^2)=a(x)\cdot a(y^2)=a(x)$. If $(x,y)\neq 1$ we can write: $x=\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}$ and $y=\prod_{k=1}^{s}q_{jk}^{\gamma_{jk}}\cdot\prod_{t=1}^{n}d_{l_t}^{\gamma_{l_t}}$ where $p_{ih}\neq d_{l_t}$, $q_{jk}\neq d_{l_t}$, $p_{ih}\neq q_{jk}$, $\forall h=\overline{1,r}$, $k=\overline{1,s}$, $t=\overline{1,n}$, but this

implies $\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}},\ \prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)=1$ and

$$\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}},\ \prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}}\right)=1\Rightarrow a(xy^2)=a\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)=$$

$$a\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}}\right)\cdot a\left(\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)=a\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\right)\cdot a\left(\prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}}\right)\cdot a\left(\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)$$

$$a\left(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\right)\cdot a\left(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)=a\left(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}\right)=a(x) \text{ because}$$

$$a\left(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)=\prod_{t=1}^{n} d_{l_t}^{\beta_{l_t}}=a\left(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}\right), \text{ where } \beta_{l_t}=\begin{cases}1 \text{ if } \alpha_{l_t}+2\gamma_{l_t} \text{ is odd}\\ 0 \text{ if } \alpha_{l_t}+2\gamma_{l_t} \text{ is even}\end{cases}$$

$$=\begin{cases}1 \text{ if } \alpha_{l_t} \text{ is odd}\\ 0 \text{ if } \alpha_{l_t} \text{ is even}\end{cases}$$

Consequence 1. For every $x\in\mathbf{N}^*$ and $n\in\mathbf{N}$, $a(x^n)=\begin{cases}1 & \text{if } n \text{ is even}\\ a(x) & \text{if } n \text{ is odd}\end{cases}$.

Consequence 2. If $\frac{x}{y}=\frac{m^2}{n^2}$ where $\frac{m}{n}$ is a simplified fraction, then $a(x)=a(y)$. It is easy to prove this, because $x=km^2$ and $y=kn^2$ and using the above property we have: $a(x)=a(km^2)=a(k)=a(kn^2)=a(y)$.

Property 7. *The sumatory numerical function of the function* a *is* $F(n)=\prod_{j=1}^{k}(H(\alpha_{i_j})(p_{i_j}+1)+\frac{1+(-1)^{\alpha_{i_j}}}{2})$ *where the prime factorization of* n *is* $n=p_{i_1}^{\alpha_{i_1}}\cdot p_{i_2}^{\alpha_{i_2}}\cdot\ldots\cdot p_{i_k}^{\alpha_{i_k}}$ *and* $H(\alpha)$ *is the number of the odd numbers which are smaller than* α.

Proof. The sumatory numerical function of a is defined as $F(n)=\sum_{d/n}a(d)$, because $(p_{i_1}^{\alpha_{i_1}},\prod_{t=2}^{k}p_{i_t}^{\alpha_{i_t}})=1$ we can use the property 4 and we obtain: $F(n)=\left(\sum_{d_1/p_{i_1}^{\alpha_{i_1}}}a(d_1)\right)\cdot\left(\sum_{d_2/p_{i_2}^{\alpha_{i_2}}\ldots p_{i_k}^{\alpha_{i_k}}}a(d_2)\right)$ and so on, making a finite number of steps we obtain $F(n)=\prod_{j=1}^{k}F(p_{i_j}^{\alpha_{i_j}})$. But we observe that

$$F(p^\alpha)=\begin{cases}\frac{\alpha}{2}(p+1)+1 & \text{if } \alpha \text{ is an even number}\\ \left(\left[\frac{\alpha}{2}\right]+1\right)(p+1) & \text{if } \alpha \text{ is an odd number}\end{cases}$$

where p is a prime number.

If we take into account of the definition of $H(\alpha)$ we find $H(\alpha)=\begin{cases}\frac{\alpha}{2} & \text{if } \alpha \text{ is even}\\ \left[\frac{\alpha}{2}\right]+1 & \text{if } \alpha \text{ is odd}\end{cases}$ so we can write $F(p^\alpha)=H(\alpha)\cdot(p+1)+\frac{1+(-1)^\alpha}{2}$,

therefore: $F(n)=\prod_{j=1}^{k}(H(\alpha_{i_j})(p_{i_j}+1)+\frac{1+(-1)^{\alpha_{i_j}}}{2}$.

In the sequel we study some equations which involve the function a.

1) Find the solutions of the equation: $xa(x)=m$, where $x,m \in \mathbf{N}^*$.

If m is not a perfect square then the above equation has not solutions.

If m is a perfect square, $m=z^2, z \in \mathbf{N}^*$, then we have to give the solutions of the equation $xa(x)=z^2$.

Let $z=p_{i_1}^{\alpha_1} \cdot p_{i_2}^{\alpha_2} \cdots p_{i_k}^{\alpha_k}$ be the prime factorization of z. Then $xa(x)=p_{i_1}^{2\alpha_1} \cdot p_{i_2}^{2\alpha_2} \cdots p_{i_k}^{2\alpha_k}$, so taking account of the definition of the function a, the equation has the following solutions: $x_1^{(0)}=p_{i_1}^{2\alpha_1} \cdot p_{i_2}^{2\alpha_2} \cdots p_{i_k}^{2\alpha_k}$ (because $a(x_1^{(0)})=1$), $x_1^{(1)}=p_{i_1}^{2\alpha_1-1} \cdot p_{i_2}^{2\alpha_2} \cdots p_{i_k}^{2\alpha_k}$ (because $a(x_1^{(1)})=p_{i_1}$), $x_2^{(1)}=p_{i_1}^{2\alpha_1} \cdot p_{i_2}^{2\alpha_2-1} \cdot p_{i_3}^{2\alpha_3} \cdots p_{i_k}^{2\alpha_k}$ (because $a(x_2^{(1)})=p_{i_2}$), ..., $x_k^{(1)}=p_{i_1}^{2\alpha_1} \cdot p_{i_2}^{2\alpha_2} \cdots p_{i_k}^{2\alpha_k-1}$ (because $a(x_k^{(1)})=p_{i_k}$), then $x_t^{(2)}=\dfrac{z^2}{p_{i_{j_1}} \cdot p_{i_{j_2}}}$, $j_1 \neq j_2$, $j_1,j_2 \in \{i_1,\dots,i_k\}$, $t=\overline{1,C_k^2}$ (because $a(x_t^{(2)})=p_{i_{j_1}} \cdot p_{i_{j_2}}$), and, in an analogue way, $x_t^{(3)}$, $t \in \overline{1,C_k^3}$ has as values $\dfrac{z^2}{p_{i_{j_1}} \cdot p_{i_{j_2}} \cdot p_{i_{j_3}}}$, where $j_1,j_2,j_3 \in \{i_1,\dots,i_k\}$ $j_1 \neq j_2, j_2 \neq j_3, j_3 \neq j_1$, and so on, $x_1^{(k)}=\dfrac{z^2}{p_{i_1} \cdot p_{i_2} \cdots p_{i_k}}=\dfrac{z^2}{z}=z$. So the above equation has $1+C_k^1+C_k^2+\dots-C_k^k=2^k$ different solutions where k is the number of the prime divisors of m.

2) Find the solutions of the equation: $xa(x)+ya(y)=za(z)$, $x,y,z \in \mathbf{N}^*$.

Proof. We note $xa(x)=m^2$, $ya(y)=n^2$ and $za(z)=s^2$, $x,y,z \in \mathbf{N}^*$ and the equation

$$m^2+n^2=s^2, \quad m,n,s \in \mathbf{N}^* \qquad (*)$$

has the following solutions: $m=u^2-v^2$, $n=2uv$, $s=u^2+v^2$, $u>v>0$, $(u,v)=1$ and u and v have different evenes.

If (m,n,s) as above is a solution, then $(\alpha m, \alpha n, \alpha s)$, $\alpha \in \mathbf{N}^*$ is also a solution of the equation (*).

If (m,n,s) is a solution of the equation (*), then the problem is to find the solutions of the equation $xa(x)=m^2$ and we see from the above problem that there are 2^{k_1} solutions (where k_1 is the number of the prime divisors of m), then the solutions of the equations $ya(y)=n^2$ and respectively $za(z)=s^2$, so the number of the different solutions of the given equations, is $2^{k_1} \cdot 2^{k_2} \cdot 2^{k_3}=2^{k_1+k_2+k_3}$ (where k_2 and k_3 have the same signifience as k_1, but concerning n and s, respectively).

For $\alpha>1$ we have $xa(x)=\alpha^2 m^2$, $ya(y)=\alpha^2 n^2$, $za(z)=\alpha^2 s^2$ and, using an analogue way as above, we find $2^{k_1 \cdot k_2 \cdot k_3}$ different solutions, where k_i, $i=\overline{1,3}$ is the number of the prime divisors of αm, αn and αs, respectively.

Remark. In the particular case $u=2$, $v=1$ we find the solution (3,4,5) for (*). So we must find the solutions of the equations $xa(x)=3^2\alpha^2$, $ya(y)=2^4\alpha^2$ and $za(z)=5^2\alpha^2$, for $\alpha \in \mathbf{N}^*$. Suppose that α has not 2,3 and 5 as prime factors in this prime factorization $\alpha=p_{i_1}^{\alpha_1} \cdot p_{i_2}^{\alpha_2} \cdots p_{i_k}^{\alpha_k}$. Then we have:

$$xa(x)=3^2\alpha^2 \Rightarrow x\in\{3^2\alpha^2,\frac{3^2\alpha^2}{p_{i1}},\dots,\frac{3^2\alpha^2}{p_{ik}},\frac{3^2\alpha^2}{p_{i1}\cdot p_{i2}},\dots,\frac{3^2\alpha^2}{p_{ik-1}\cdot p_{ik}},\dots,\frac{3^2\alpha^2}{p_{i1}\cdots p_{ik-1}},\dots,\frac{3^2\alpha^2}{p_{i2}\cdots p_{ik}},$$

$$3^2\alpha\ ,3\ \alpha^2,\frac{3\ \alpha^2}{p_{i1}},\dots,\frac{3\ \alpha^2}{p_{ik}},\frac{3\ \alpha^2}{p_{i1}\cdot p_{i2}},\dots,\frac{3\ \alpha^2}{p_{ik-1}\cdot p_{ik}},\dots,\frac{3\ \alpha^2}{p_{i1}\cdots p_{ik-1}},\dots,\frac{3\ \alpha^2}{p_{i2}\cdots p_{ik}},3\alpha\}$$

$$ya(y)=4^2\alpha^2 \Rightarrow y\in\{4^2\alpha^2,\frac{4^2\alpha^2}{p_{i1}},\dots,\frac{4^2\alpha^2}{p_{ik}},\frac{4^2\alpha^2}{p_{i1}\cdot p_{i2}},\dots,\frac{4^2\alpha^2}{p_{ik-1}\cdot p_{ik}},\dots,\frac{4^2\alpha^2}{p_{i1}\cdots p_{ik-1}},\dots,\frac{4^2\alpha^2}{p_{i2}\cdots p_{ik}},$$

$$4^2\alpha\ ,8\ \alpha^2,\frac{8\ \alpha^2}{p_{i1}},\dots,\frac{8\ \alpha^2}{p_{ik}},\frac{8\ \alpha^2}{p_{i1}\cdot p_{i2}},\dots,\frac{8\ \alpha^2}{p_{ik-1}\cdot p_{ik}},\dots,\frac{8\ \alpha^2}{p_{i1}\cdots p_{ik-1}},\dots,\frac{8\ \alpha^2}{p_{i2}\cdots p_{ik}},8\alpha\}$$

$$za(y)=5^2\alpha^2 \Rightarrow z\in\{5^2\alpha^2,\frac{5^2\alpha^2}{p_{i1}},\dots,\frac{5^2\alpha^2}{p_{ik}},\frac{5^2\alpha^2}{p_{i1}\cdot p_{i2}},\dots,\frac{5^2\alpha^2}{p_{ik-1}\cdot p_{ik}},\dots,\frac{5^2\alpha^2}{p_{i1}\cdots p_{ik-1}},\dots,\frac{5^2\alpha^2}{p_{i2}\cdots p_{ik}},$$

$$5^2\alpha\ ,5\ \alpha^2,\frac{5\ \alpha^2}{p_{i1}},\dots,\frac{5\ \alpha^2}{p_{ik}},\frac{5\ \alpha^2}{p_{i1}\cdot p_{i2}},\dots,\frac{5\ \alpha^2}{p_{ik-1}\cdot p_{ik}},\dots,\frac{5\ \alpha^2}{p_{i1}\cdots p_{ik-1}},\dots,\frac{5\ \alpha^2}{p_{i2}\cdots p_{ik}},5\alpha\}$$

So any triplet (x_0,y_0,z_0) with x_0,y_0 and z_0 arbitrary of above corresponding values, is a solution for the equation (for example (9,16,25), is a solution).

Definition. The triplets which are the solutions of the equation $xa(x)+ya(y)=za(z)$, $x,y,z\in\mathbf{Z}^*$ we call MIV numbers.

3) Find the natural numbers x such that $a(x)$ is a three - cornered, a squared and a pentagonal number.

Proof. Because 1 is the only number which is at the same time a three - cornered, a squared and a pentagonal number, then we must find the solutions of the equation $a(x)=1$, therefore x is any perfect square.

4) Find the solutions of the equation: $\frac{1}{xa(x)}+\frac{1}{ya(y)}=\frac{1}{za(z)}$, $x,y,z\in\mathbf{N}^*$.

Proof. We have $xa(x)=m^2, ya(y)=n^2, za(z)=s^2$, $m,n,s\in\mathbf{N}^*$.

The equation $\frac{1}{m^2}+\frac{1}{n^2}=\frac{1}{s^2}$ has the solutions:

$$m=t(u^2+v^2)2uv$$

$$n=t(u^2+v^2)(u^2-v^2)$$

$$s=t(u^2-v^2)2uv,$$

$u>v$, $(u, v)=1$, u and v have different eveness and $t \in \mathbf{N}^*$, so we have

$$xa(x) = t^2(u^2+v^2)^2 4u^2v^2$$
$$ya(y) = t^2(u^2+v^2)^2(u^2-v^2)^2$$
$$za(z) = t^2(u^2-v^2)^2 4u^2v^2$$

and we find x, y and z in the same way which is indicated in the first problem.

For example, if $u=2$, $v=1$, $t=1$ we have $m=20$, $n=15$, $s=12$, so we must find the solutions of the following equations:

$$xa(x) = 20^2 = 2^4 \cdot 5^2 \Rightarrow x \in \{2^3 \cdot 5^2 = 200,\ 2^4 \cdot 5 = 80,\ 2^3 \cdot 5 = 40,\ 2^4 \cdot 5^2 = 400\}$$

$$ya(y) = 15^2 = 3^2 \cdot 5^2 \Rightarrow y \in \{15, 45, 75, 225\}$$

$$za(z) = 12^2 = 2^4 \cdot 3^2 \Rightarrow z \in \{24, 48, 72, 144\}$$

Therefore for this particular values of u, v and t we find $4 \cdot 4 \cdot 4 = 2^2 \cdot 2^2 \cdot 2^2 = 2^6 = 64$ solutions. (because $k_1 = k_2 = k_3 = 2$)

5) Find the solutions of the equation: $a(x)+a(y)+a(z) = a(x)a(y)a(z)$, $x,y,z \in \mathbf{N}^*$.

Proof. If $a(x)=m$, $a(y)=n$ and $a(z)=s$, the equation $m+n+s = m \cdot n \cdot s$, $m,n,s \in \mathbf{N}^*$ has a solutions the permutations of the set $\{1,2,3\}$ so we have:

$a(x) = 1 \Rightarrow x$ must be a perfect square, therefore $x = u^2$, $u \in \mathbf{N}^*$
$a(y) = 2 \Rightarrow y = 2v^2$, $v \in \mathbf{N}^*$
$a(z) = 3 \Rightarrow z = 3t^2$, $t \in \mathbf{N}^*$.

Therefore the solutions are the permutation of the sets $\{u^2, 2v^2, 3t^2\}$ where $u,v,t \in \mathbf{N}^*$.

6) Find the solutions of the equation $Aa(x)+Ba(y)+Ca(z)=0$, $A,B,C \in \mathbf{Z}^*$.

Proof. If we note $a(x)=u, a(y)=v, a(z)=t$ we must find the solutions of the equation $Au+Bv+Ct=0$.

Using the method of determinants we have:

$$\begin{vmatrix} A & B & C \\ A & B & C \\ m & n & s \end{vmatrix} = 0, \quad \forall m,n,s \in \mathbf{Z} \Rightarrow A(Bs-Cn)+B(Cm-As)+C(An-Bm)=0,$$ and it is known that the only solutions are

$$u = Bs - Cn$$
$$v = Cm - As$$
$$t = An - Bm, \quad \forall m,n,s \in \mathbf{Z}$$

so, we have

$$a(x) = Bs - Cn$$
$$a(y) = Cm - As$$
$$a(z) = An - Bm$$

and now we know to find x, y and z.

Example. If we have the following equation: $2a(x)-3a(y)-a(z)=0$, usind the above result we must find (with the above mentioned method) the solutions of the equations:

$a(x) = -3s + n$
$a(y) = -m - 2s$
$a(z) = 2n + 3m$, m, n and $s \in \mathbf{Z}$.

For $m = -1$, $n = 2$, $s = 0$: $a(x) = 2$, $a(y) = 1$, $a(z) = 1$ so, the solution in this case is $(2\alpha^2, \beta^2, \gamma^2)$, $\alpha, \beta, \gamma \in \mathbf{Z}^*$. For the another values of m, n, s we find the corresponding solutions.

7) The same problem for the equation $Aa(x) + Ba(y) = C$, $A, B, C \in \mathbf{Z}$.

Proof. $Aa(x) + Ba(y) - C = 0 \Leftrightarrow Aa(x) + Ba(y) + (-C)a(z) = 0$ with $a(z) = 1$ so we must have $An - Bm = 1$. If n_0 and m_0 are solutions of this equation $(An_0 - Bm_0 = 1)$ it remains us to find the solutions of the following equations:

$a(x) = Bs + Cn_0$
$a(y) = -Cm_0 - As$, $s \in \mathbf{Z}$, but we know how to find them.

Example. If we have the equation $2a(x) - 3a(y) = 5$, $x, y \in \mathbf{N}^*$ using the above results, we get: A=2, $B = -3$, $C = -5$ and $a(z) = 1 = 2n + 3m$. The solutions are $m = 2k + 1$ and $n = -1 - 3k$, $k \in \mathbf{Z}$. For the particular value $k = -1$ we have $m_0 = -1$ and $n_0 = 2$ so we find
$a(x) = -3 + 5 \cdot 2 = 10 - 3s$ and
$a(y) = -5(-1) - 2s = 5 - 2s$.

If $s_0 = 0$ we find $a(x) = 10 \Rightarrow x = 10u^2$, $u \in \mathbf{Z}^*$
$a(y) = 5 \Rightarrow y = 5v^2$, $v \in \mathbf{Z}^*$ and so on.

8) Find the solutions of the equation: $a(x) = ka(y)$ $k \in \mathbf{N}^*$ $k > 1$.

Proof. If k has in his prime factorization a factor which has an exponent ≥ 2, then the problem has not solutions.

If $k = p_{i_1} \cdot p_{i_2} \cdots p_{i_r}$ and the prime factorizarion of $a(y)$ is $a(y) = q_{j_1} \cdot q_{j_2} \cdots q_{j_u}$, then we have solutions only in the case $p_{i_1}, p_{i_2}, \dots p_{i_r} \notin \{q_{j_1}, q_{j_2}, \dots, q_{j_u}\}$.

This implies that $a(x) = p_{i_1} \cdot p_{i_2} \cdots p_{i_r} \cdot q_{j_1} \cdot q_{j_2} \cdots q_{j_u}$, so we have the solutions

$$x = p_{i1} \cdot p_{i2} \cdots p_{ir} \cdot q_{j1} \cdot q_{j2} \cdots q_{ju} \cdot \alpha^2$$
$$y = q_{j_1} \cdot q_{j_2} \cdots q_{j_u} \cdot \beta^2, \ \alpha, \beta \in \mathbf{Z}^*.$$

9) Find the solutions of the equation $a(x)=x$ (the fixed points of the function a).

Proof. Obviously, $a(1)=1$. Let $x>1$ and let $x = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}}$, $\alpha_{i_j} \geq 1$, for $j = \overline{1,r}$ be the prime factorization of x. Then $a(x) = p_{i_1}^{\beta_{i_1}} \cdot p_{i_2}^{\beta_{i2}} \cdots p_{i_r}^{\beta_{i_r}}$ and $\beta_{i_j} \leq 1$ for $j = \overline{1,r}$. Because $a(x)=x$ this implies that $\alpha_{i_j} = \beta_{i_j} = 1$, $\forall j \in \overline{1,r}$, therefore $x = p_{i_1} \cdot p_{i_2} \cdots p_{i_r}$, where p_{i_j}, $j = \overline{1,r}$ are prime numbers.

REFERENCES

F. SMARANDACHE *Only problems, not solutions !*, Xiquan Publishing House, Phoenix - Chicago, 1990, 1991, 1993.

SOME REMARKS CONCERNING THE DISTRIBUTION OF THE SMARANDACHE FUNCTION

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The Smarandache function is a numerical function $S:N^* \to N^*$ $S(k)$ representing the smallest natural number n such that n! is divisible by k. From the definition it results that $S(1)=1$.

I will refer for the beginning the following problem:

"Let k be a rational number, $0 < k \leq 1$. Does the diophantine equation $\frac{S(n)}{n} = k$ has always solutions? Find all k such that the equation has an infinite number of solutions in N^*" from "Smarandache Function Journal".

I intend to prove that equation hasn't always solutions and case that there are an infinite number of solutions is when $k = \frac{1}{r}$, $r \in N^*$, $k \in Q$ and $0 < k \leq 1 \Rightarrow$ there are two relatively prime non negative integers p and q such that $k = \frac{q}{p}$, $p,q \in N^*$, $0 < q \leq p$. Let n be a solution of the equation $\frac{S(n)}{n} = k$. Then $\frac{S(n)}{n} = \frac{p}{q}$, (1). Let d be a highest common divisor of n and S(n) : $d = (n, S(n))$. The fact that p and q are relatively prime and (1) implies that $S(n) = qd$, $n = pd \Rightarrow S(pd) = qd$ (*).

This equality gives us the following result: $(qd)!$ is divisible by $pd \Rightarrow [(qd - 1)! \cdot q]$ is divisible by p. But p and q are relatively prime integers, so $(qd-1)!$ is divisible by p. Then $S(p) \leq qd - 1$.

I prove that $S(p) \geq (q - 1)d$.

If we suppose against all reason that $S(p) < (q - 1)d$, it means $[(q - 1)d - 1]!$ is divisible by p. Then $(pd) \mid [(q - 1)d]!$ because $d \mid (q - 1)d$, so $S(pd) \leq (q - 1)d$. This is contradiction with the fact that $S(pd) = qd > (q - 1)d$. We have the following inequalities:

$(q - 1)d \leq S(p) \leq qd - 1$.

For $q \geq 2$ we have from the first inequality $d \leq \frac{S(p)}{q-1}$ and from the second $\frac{S(p+1)}{q} \leq d$, so $\frac{S(p+1)}{q} \leq d \leq \frac{S(p)}{q-1}$.

For $k=\frac{q}{p}$, q ≥ 2, the equations has solutions if and only if there is a natural number between $\frac{S(p+1)}{q}$ and $\frac{S(p)}{q-1}$. If there isn't such a number, then the equation hasn't solutions. However, if there i a number d with $\frac{S(p+1)}{q} \le d \le \frac{S(p)}{q-1}$, this doesn't mean that the equation has solutions. This condition is necessary but not sufficient for the equation to have solutions.

For example:

a) $k=\frac{4}{5}$, q =4 , p=5 $\Rightarrow$ $\frac{S(p+1)}{q}=\frac{6}{4}=\frac{3}{2}$, $\frac{S(p)}{q-1}=\frac{5}{3}$. In this case the equation hasn't solutions.

b) $k=\frac{3}{10}$, q=3, p=10 ; S(10)=5 , $\frac{6}{3}=2 \le d \le \frac{5}{2}$. If the equation has solutions, then we must have d=2 , n=dp=20 , S(n)=dq=6. But S(20)=5.

This is a contradiction. So there are no solutions for $h=\frac{3}{10}$.

We can ha·e more then natural numbers between $\frac{S(p+1)}{q}$ and $\frac{S(p)}{q-1}$. For example:

$k=\frac{3}{29}$, q=3 , p=29, $\frac{S(p+1)}{q}=10$, $\frac{S(p)}{q-1}=14{,}5$.

We prove that the equation $\frac{S(n)}{n}=k$ hasn't always solutions.

If q ≥ 2 then the number of solutions is equal with the number of values of d that verify relation (*). But d can be a nonnegative integer between $\frac{S(p+1)}{q}$ and $\frac{S(p)}{q-1}$, so d can take only a finite set of values. This means that the equation has no solutions or it has only a finite number of solutions.

We study note case $k=\frac{1}{p}$, p ∈ N*. In this case he equation has an infinite number of solutions. Let p_0 be a prime number such that $p<p_0$ and $n=pp_0$. We have $S(n)=S(pp_0)=p$, so $S(n)=p_0$. $\frac{S(n)}{n}=\frac{p_0}{pp_0}=\frac{1}{p}$, so the equation has an infinite number of solution.

I will refer now to another problem concerning the ratio $\frac{S(n)}{n}$ "Is there an infinity of natural numbers such that $0<\left\{\frac{x}{S(x)}\right\}<\left\{\frac{S(x)}{x}\right\}$?" from the same journal.

I will prove that the only number x that verifies the inequalities is x=9 : S(9)=6, $\frac{S(x)}{x}=\frac{6}{9}=\frac{2}{3}$, $\left\{\frac{x}{S(x)}\right\}=\left\{\frac{9}{6}\right\}=\frac{1}{2}$ and $0<\frac{1}{2}<\frac{2}{3}$, so x=9 verifies $0<\left\{\frac{x}{S(x)}\right\}<\left\{\frac{S(x)}{x}\right\}$.

Let $x=p_1^{\alpha_1}\dots p_n^{\alpha_n}$ be the standard form of x.

$S(x)=\max\limits_{1\le k\le n} S(p_k^{\alpha_k})$. We put $S(x)=S(p^{\alpha})$, where p^{α} is one of $p_1^{\alpha_1}\dots p_n^{\alpha_n}$ such that $S(p^{\alpha})=\max\limits_{1\le k\le n} S(p_k^{\alpha_k})$.

$\left\{\frac{x}{S(x)}\right\}$ can take one of the following values : $\frac{1}{S(x)}$, $\frac{2}{S(x)}$, ... , $\frac{S(x)-1}{S(x)}$ because $0 < \left\{\frac{x}{S(x)}\right\} < \left\{\frac{S(x)}{x}\right\}$ (We have $S(x) \le x$, so $\frac{S(x)}{x} \le 1$ and $\left\{\frac{S(x)}{x}\right\} \le \frac{S(x)}{x}$). This means $\frac{S(x)}{x} \ge \frac{1}{S(x)} \Rightarrow S(p^{\alpha})^2 > x \ge p^{\alpha}$. (2)

But $(\alpha p)! = 1\cdot 2\cdot \ldots \cdot p(p+1)\ldots(2p)\ldots(\alpha p)$ is divisible by p^{α} , so $\alpha p \ge S(p^{\alpha})$. From this last inequality and (2) it follows that $\alpha^2p^2>p^2$. We have three cases:

I. $\alpha=1$. In this case $S(x)=S(p)=p$, x is divisible by p, so $\frac{x}{p} \in Z$. This is a contradiction. There are no solutions for $\alpha=1$.

II. $\alpha=2$. In this case $S(x)=S(p^2)=2p$, because p is a prime number and $(2p)! = 1\cdot 2\cdot \ldots \cdot p(p+1)\ldots(2p)$, so $S(p^2)=2p$.

But $\left\{\frac{px_1}{2}\right\} \in \left\{0, \frac{1}{2}\right\}$. This means $\left\{\frac{px_1}{2}\right\} = \frac{1}{2} \Rightarrow \frac{1}{2} < \frac{2}{px_1} < 4$; p is a prime number $\Rightarrow p \in \{2,3\}$.

If p=2 and $px_1 < 4 \Rightarrow x_1 = 1$, but x=4 isn't a solution of the equation: $S(4)=4$ and $\left\{\frac{4}{4}\right\} = 0$.

If p=3 and $px_1 < 4 \Rightarrow x_1 = 1$. so $x=p^2=9$ is a solution of equation.

III. $\alpha=3$. We have $\alpha^2p^2>p^{\alpha} \Leftrightarrow \alpha^2 > p^{\alpha-1}$.

For $\alpha \ge 8$ we prove that we have $p^{\alpha-2}>p^2$, $(\forall)\, p \in N^*$, $p \ge 2$.

We prove by induction that $2^{n-1} > (n+1)^2$.

$2^{n-1} = 2\cdot 2^{n-2} \ge 2\cdot n^2 = n^2+n^2 \ge n^2+8n > n^2+2n+1 = (n+1)^2$, because $n \ge 8$.

We proved that $p^{\alpha-2} \ge 2^{\alpha-1} \ge \alpha^2$, for any $\alpha \ge 8$, $p \in N^*$, $p \ge 2$.

We have to study the case $\alpha \in \{ 3,4,5,6,7\}$.

a) $\alpha=3 \Rightarrow p \in \{ 2,3,5,7\}$, because p is a prime number.

If p=2 then $S(x)=S(2^3)=4$. But x is divisible by 8, so $\left\{\frac{x}{S(x)}\right\} = \left\{\frac{x}{4}\right\} = 0$, so x=4 cannot be a solution of the inequation.

If $p=3 \Rightarrow S(x)=S(3^3)=9$. ~~But x is divisible by 27~~, so $\left\{\frac{x}{S(x)}\right\} = \left\{\frac{x}{9}\right\} = 0$, so x=9 cannot be a solution of the inequation.

If $p=5 \Rightarrow S(x)=S(5^3)=15$; $\left\{\frac{x}{S(x)}\right\} = \left\{\frac{S(x)}{x}\right\} = 0$ $x=5^3\cdot x_1$, $x_1 \in N^*$, $(5,x_1)=1$.

We have $0<\left\{\frac{5^2\cdot x_1}{3}\right\}<\left\{\frac{3}{5^2\cdot x_1}\right\}$. This first inequality implies $\left\{\frac{5^2\cdot x_1}{3}\right\}\in\left\{\frac{1}{3},\frac{2}{3}\right\}$, so $\frac{1}{3}<\frac{3}{5^2\cdot x_1}\Rightarrow 5^2\cdot x_1<9$, but this is impossible.

If $p=7\Rightarrow S(x)=S(7^3)=21$, $x=7^3\cdot x_1$, $(7,x_1)=1$, $x1\in N^*$.

We have $0<\left\{\frac{x}{S(x)}\right\}<\left\{\frac{S(x)}{x}\right\}\Rightarrow 0<\left\{\frac{7^2\cdot x_1}{3}\right\}<\frac{3}{7^2\cdot x_1}$. But $0<\left\{\frac{7^2\cdot x_1}{3}\right\}$ implies $\left\{\frac{7^2\cdot x_1}{3}\right\}\in\left\{\frac{1}{3},\frac{2}{3}\right\}$.

W have $\frac{1}{3}\le\left\{\frac{7^2\cdot x_1}{3}\right\}\Rightarrow 7^2\cdot x_1<9$, but is impossible.

b) $\alpha=4$: $16\Rightarrow p\in\{2,3\}$.

If $p=2\Rightarrow S(x)=S(x^2)=6$, $x=16\cdot x_1$, $x_1\in N^*$, $(2,x_1)=1$, $0<\left\{\frac{x}{S(x)}\right\}<\frac{S(x)}{x}\Rightarrow 0<\left\{\frac{8x_1}{3}\right\}<\frac{3}{8x_1}$.

$0<\left\{\frac{8x_1}{3}\right\}\Rightarrow x_1=1\Rightarrow x=16$.

But $\frac{S(x)}{x}=\frac{6}{16}=\frac{3}{8}$; $\left\{\frac{x}{S(x)}\right\}=\left\{\frac{16}{6}\right\}=\left\{\frac{8}{3}\right\}=\frac{2}{3}\cdot\frac{2}{3}>\frac{3}{8}$, so the inequality isn't verified.

If $p=3\Rightarrow S(x)=S(3^4)=9$, $x=3^4\cdot x_1$, $(3,x_1)=1\Rightarrow 9|x\Rightarrow\frac{x}{S(x)}=0$, so the inequality isn't verified.

For $\alpha=\{5,6,7\}$, the only natural number $p>1$ that verifies the inequality $\alpha^2>p^{\alpha-2}$ is 2:

$\alpha=5$:$25>p^3\Rightarrow p=2$

$\alpha=6$: $36>p^4\Rightarrow p=2$

$\alpha=7$: $49>p$

In every case $x=2^\alpha\cdot x_1$, $x_1\in N^*$, $(x_1,2)=1$, and $S(x_1)\le S(2^\alpha)$.

But $S(2^5)=S(2^6)=S(2^7)8$, so $S(x)=8$ But x is divisible by 8, so $\left\{\frac{x}{S(x)}\right\}=0$ so the inequality isn't verified because $0=\left\{\frac{x}{S(x)}\right\}$. We found that there is only x=9 to verify the inequality $0<\left\{\frac{x}{S(x)}\right\}<\left\{\frac{S(x)}{x}\right\}$

I try to study some diophantine equations proposed in "Smarandache Function Journal".

1) I study the equation $S(mx)=mS(x)$, $m\ge2$ and x is a natural number.

Let x be a solution of the equation.

We have $S(x)!$ is divisible by x It is known that among m consecutive numbers, one is divisible by m, so $(S(x)!)$is divisible by m, so $(S(x)+1)(S(x+2)...(S(x)+m)$ is divisible by (mx). We know that $S(mx)$ is the smallest natural number such that $S(mx)!$ is divisible by (mx) and this implies $S(mx)\leq S(x)+m$. But $S(mx)=mS(x)$, so $mS(x)\leq S(x)+m \Leftrightarrow mS(x)-S(x)-m+1 \leq \Leftrightarrow (m-1)(S(x)-1)\leq 1$. We have several cases:

If m=1 then the equation becomes $S(X)=S(x)$, so any natural number is a solution of the equation.

If m=2, we have $S(x) \in \{1,2\}$ implies $x \in \{1,2\}$. We conclude that if m=1 then any natural number is a solution of the equation of the equation; if m=2 then x=1 and x=2 are only solution and if $m \geq 3$ the only solution of the equation is x=1.

2) Another equation is $S(x^y)=y^x$, x, y are natural numbers.

Let (x,y) be a solution of the equation.

$(yx)!=1...x(x+1)...(2x)...(yx)$ implies $S(x^y) \leq yx$, so $y^x\leq yx_1$ because $S(x^y)=y^x$.

But $y \geq 1$, so $y^{x-1}\leq x$.

If x=1 then equation becomes $S(1) = y$, so y=1, so x=y=1 is a solution of the equation.

If $x\geq 2$ then $x\geq 2^{x-1}$. But the only natural numbers that verify this inequality are x=y=2:

x=y=2 verifies the equation, so x=y=2 is a solution of the equation.

For $x\geq 3$ we prove that $x<2^{x-1}$.We make the proof by induction.

If x=3 : $3<2^{3-1}=4$.

We suppose that $k<2^{k-1}$ and we prove that $k+1<2^k$.We have $2^k=2\cdot 2^k>2\cdot k=k+k>k+1$, so the inequality is established and there are no other solutions then x=y=1 and x=y=2.

3) I will prove that for any m,n natural numbers, if m>1 then the equation $S(x^n)=x^m$ has no solution or it has a finite number of solutions, and for m=1 the equation has a infinite number of solutions.

I prove that $S(x^n)\leq nx$. But $x^m=S(x^n)$, so $x^m \leq nx$.

For $m\geq 2$ we have $x^{m-1} \leq n$. If m=2 then $x\leq n$, and if $m \geq 3$ then $x \leq \sqrt[m-1]{n}$, so x can take only a finite number of values, so the equation can have only a finite number of solutions or it has no solutions.

We notice that x=1 is a solution of the equation for any m,n natural numbers.

If the equation has a solution different of 1, we must have $x^m=S(x^n) \leq x^n$., so $m \leq n$

If m=n, the equation becomes $x^{m=n}=S(x^n)$, so x^n is a prime number or $x^n=4$, so n=1 and any prime number as well as x=4 is a solution of the equation, or n=2 and the only solutions are x=1 and x=2.

For m=1 and $n \geq 1$, we prove that the equations $S(x^m)=x$, $x \in N^*$ has an infinite number of solutions. Let be a prime number, p>n. We prove that)np) is a solution of the equation, that is $S((np)^n)=np$.

n<p and p is a prime number, so n and p are relatively prime numbers.

n<p implies:

$(np)! = 1 \cdot 2 \cdot \ldots \cdot n(n+1) \cdot \ldots \cdot (2n) \cdot \ldots \cdot (pn)$ is divisible by n^n.

$(np)! = 1 \cdot 2 \cdot \ldots \cdot p(p+1) \cdot \ldots \cdot (2p) \cdot \ldots \cdot (pn)$ is divisible by p^n.

But p and n are relatively prime numbers, so (np)! is divisible by $(np)^n$.

If we suppose that $S((np)^n)<np$, then we find that (np-1)! is a divisible by $(np)^n$, so(np-1)1 is divisible by p^n(3). But the exponent of p in the standard form of p in the standard form of (np-1)! is:

$$E=\left[\frac{np-1}{p}\right]+\left[\frac{np-1}{p^2}\right]+\ldots$$

But p >n, so p^2 >np >np-1. This implies :

$\left[\frac{np-1}{p^k}\right]=0$, for any $k \geq 2$. We have:

$$E=\left[\frac{np-1}{p}\right]=n-1 .$$

This means (np-1)! is divisible by p^{n-1} , but isn't divisible by p^n , so this is a contradiction with (3). We proved that S((np)n)=np, so the equation S(xn)=x has an infinite number of solutions for any natural number n.

REFERENCE

Smarandache Function Journal, Vol.1 , 1990, Number Theory Publishing, Co,

R. Muller Editor, Phoenix, New York, Lyon.

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SOME ELEMENTARY ALGEBRAIC CONSIDERATIONS INSPIRED BY THE SMARANDACHE FUNCTION

by

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It is known that the Smarandache function $S:N^*\dashrightarrow N^*$, $S(n)=\min\{k \mid n \text{ divides } k!\}$ satisfies

(i) S is surjective

(ii) $S([m,n]) = \max\{S(m), S(n)\}$, where $[m,n]$ is the smallest common multiple of m and n.

That is on N^* there are considered both of the divisibility order $\leq_d$ ($m \leq_d n$ if and only if m divide n) and the usual order $\leq$. Of course the algebraic usual operations "+" and "·" play also an important role in the description of the properties of S.
For instance it is said that [1]:

$$\max\{S(k^k), S(n^n)\} \leq S((kn)^{kn}) \leq nS(k^k)+kS(n^n).$$

If we consider the universal algebra (N^*,Ω), with $\Omega=\{V_d, \Phi_0\}$, where $V_d:(N^*)^2\dashrightarrow N^*$ is given by, $m\ V_d\ n=[m,n]$, and $\Phi_0:(N^*)^0\dashrightarrow N^*$, is given by $\Phi_0(\{\Phi\})=1=e_{V_d}$ and analogously the universal algebra (N^*,Ω') with $\Omega'=\{V,\Psi_0\}$, where $V:(N^*)^2\dashrightarrow N^*$, is

defined by $m \vee n = \max\{m,n\}$, and $\Psi_0 : (\mathbf{N}^*)^0 \dashrightarrow \mathbf{N}^*$ is defined by $\Psi_0(\{\Phi\}) = 1 = e_\vee$, then it results:

1. PROPOSITION. Let $\overline{\mathbf{N}} = \{S^-(k) \mid k \in \mathbf{N}^*\}$, where $S^-(k) = \{x \in \mathbf{N}^* \mid S(x) = k\}$. Then

(a) $\overline{\mathbf{N}}$ is countable $(card\mathbf{N}^* = alef\ zero)$.

(b) on $\overline{\mathbf{N}}$ may be defined an universal algebra, isomorfe with $(\mathbf{N}^*, \Omega')$.

Proof. (b) Let $\omega : (\overline{\mathbf{N}})^2 \dashrightarrow \overline{\mathbf{N}}$ be defined by $\omega(S^-(a), S^-(b)) = S^-(c)$, where $c = S(x \vee_d y)$, with $x \in S^-(a)$, $y \in S^-(b)$.

Then ω is well defined because if $x_1 \in S^-(a)$, $y_1 \in S^-(b)$ the

$S(x_1 \vee_d y_1) = S(x_1) \vee S(y_1) = a \vee b = S(x) \vee S(y) = S(x \vee_d y) = c$.

Example. $\omega(S^-(23), S^-(14)) = S^-(23)$ because if for instance $x = 46 \in S^-(23)$ and $y = 49 \in S^-(14)$ then $46\ V_d 49 = 2254$ and $S(2254) = 23$.

In fact, because $c = S(x \vee_d y) = S(x) \vee S(y) = a \vee b$, it results that ω is defined by

$$\omega(S^-(a), S^-(b)) = S^-(a \vee b).$$

We define now $\omega_0 : (\overline{\mathbf{N}})^0 \dashrightarrow \overline{\mathbf{N}}$ by $\omega_0(\{\Phi\}) = S^-(1)$.

Let us note $S^-(1) = e_\omega$. Then

$$\forall S^-(k) \in \overline{\mathbf{N}} \quad \omega(S^-(k), e_\omega) = \omega(e_\omega, S^-(k)) = S^-(k).$$

Then $(\overline{\mathbf{N}}, \overline{\Omega})$ is an universal algebra if $\overline{\Omega} = \{\omega, \omega_0\}$.

It may be defined $h : \overline{\mathbf{N}} \dashrightarrow \mathbf{N}^*$ an isomorphism between $(\overline{\mathbf{N}}, \overline{\Omega})$ and $(\mathbf{N}^*, \Omega')$, by $h(S^-(k)) = k$.

We have

$$\forall S^{-}(a), S^{-}(b) \in \overline{\mathbf{N}} \quad h(\omega(S^{-}(a), S^{-}(b))) = h(S^{-}(a \vee b) =$$

$$= a \vee b = h(S^{-}(a)) \vee h(S^{-}(b))$$

that is h is a morphism.

Of course $h(\omega_0(\{\Phi\})) = \Psi_0(\{\Phi\})$ and **h** is injective.

Indeed, $h(S^{-}(a)) = h(S^{-}(b)) \Leftrightarrow a = b$ and then

$x \in S^{-}(a) \Leftrightarrow S(x) = a = b \Rightarrow x \in S^{-}(b) \Rightarrow S^{-}(a) \subset S^{-}(b)$ and analogously

$S^{-}(b) \subset S^{-}(a)$, so $S^{-}(a) = S^{-}(b)$.

From the surjectivity of S it results that **h** is surjective, because for every $k \in \mathbf{N}^{*}$ it exists $x \in \mathbf{N}^{*}$ such that $S(x) = k$, so $S^{-}(k) \neq \Phi$ and $h(S^{-}(k)) = k$.

Then we have $(\overline{\mathbf{N}}, \overline{\Omega}) \simeq (\mathbf{N}^{*}, \Omega')$ and from the bijectivity of **h** it results $card\,\overline{\mathbf{N}} = card\,\mathbf{N}^{*}$, that is the assertion (a).

Remarks (i) An other proof of Proposition 1 may be made as follows;

Let ρ_s be the equivalence associated with the function S

$$x \rho_s y \Leftrightarrow S(x) = S(y).$$

Because S is a morphism between $(\mathbf{N}^{*}, \Omega)$ and $(\mathbf{N}^{*}, \Omega')$ it results that ρ_s is a congruence and so we can define on $\dfrac{\mathbf{N}^{*}}{\rho_s}$ the operations

ω and ω_0 by

$\omega : (\mathbf{N}^*/\rho_S)^2 \dashrightarrow \mathbf{N}^*/\rho_S, \quad \omega(\hat{x}, \hat{y}) = x V_d y;$

$\omega_0 : (\mathbf{N}^*/\rho_S)^2 \dashrightarrow \mathbf{N}^*/\rho_S, \quad \omega_0(\{\Phi\}) = \hat{1}.$

Moreover, $\mathbf{N}^*/\rho_S = \overline{\mathbf{N}}$ and so it is constructed the universal algebra $(\overline{\mathbf{N}}, \overline{\Omega})$, with $\overline{\Omega} = \{\omega, \omega_0\}$. That because $S: (\mathbf{N}^*, \Omega) \dashrightarrow (\mathbf{N}^*, \Omega')$ is a morphism so by a well known isomorphism theorem it results that $(\mathbf{N}^*/\rho_S) \simeq ImS$ so $(\overline{\mathbf{N}}, \overline{\Omega}) \simeq (\mathbf{N}^*, \Omega')$. That is we have a proof for (b), the morphism being $\alpha : \overline{\mathbf{N}} \dashrightarrow \mathbf{N}^*, \alpha(\hat{x}) = S(x)$.

(ii) Proposition 1 is an argument to consider the functions

$S^{-1}_{\min} : \mathbf{N}^* \dashrightarrow \mathbf{N}^*, \quad S^{-1}_{\min}(k) = \min S^-(k)$

$S^{-1}_{\max} : \mathbf{N}^* \dashrightarrow \mathbf{N}^*, \quad S^{-1}_{\max}(k) = \max S^-(k)$ (sec [4])

whose properties we shall present in a future note.

(iii) The graph

$$G = \{(x, y) \in \mathbf{N}^* X \mathbf{N}^* \ / \ y = S(x)\}$$

is a subalgebra of the universal algebra $(\mathbf{N}^* X \mathbf{N}^*, \Omega)$, where $\Omega = \{\omega, \omega_0\}$, with $\omega : (\mathbf{N}^* X \mathbf{N}^*)^2 \dashrightarrow \mathbf{N}^* X \mathbf{N}^*$, defined by $\omega((x_1, y_1), (x_2, y_2)) = (x_1 V d x_2, y_1 \vee y_2)$ and $\omega_0 : (\mathbf{N}^* x \mathbf{N}^*)^0 \dashrightarrow \mathbf{N}^* x \mathbf{N}^*$, defined by $\omega_0(\{\Phi\}) = (\phi_0(\{\Phi\}), \Psi_0(\{\Phi\})) = (1, 1)$.

Indeed G is a subalgebra of the universal algebra $(\mathbf{N}^* x \mathbf{N}^*, \Omega)$ if for every $(x_1, y_1), (x_2, y_2) \in G$ it results $\omega((x_1, y_1), (x_2, y_2) \in G$ and $\omega_0(\{\Phi\}) \in G$. But

$\omega((x_1, y_1), (x_2, y_2)) = (x_1 V_d x_2, y_1 \vee y_2) = (x_1 V_d x_2, S(x_1) \vee S(x_2)) = (x_1 V_d x_2$

and $\omega_0(\{\Phi\}) \in G$ if and only if $(1,1) \in G$.

That is $(1, S(1)) \in G$.

In fact the algebraic property is more complete in the sense that $f: A \dashrightarrow B$ is a morphism between the universal algebras (A, Ω) and (B, Ω) of the some kind τ if and only if the graph F of the functional relation f is a subalgebra of the universal algebra $(A x B, \Omega)$.

Then the importance of remark (iii) consist in the fact that it is possible to underline some properties of the Smarandache function starting from the above mentioned subalgebra of the universal algèbra $(\mathbf{N}^* x \mathbf{N}^*, \Omega)$.

References

1. I. Bălăcenoiu: Smarandache Numeric Functions (Smarandache Function J, V. 4-5, No.1, (1994), 6-13).
2. I. Purdea, Gh. Pic: Tratat de algebră modernă, V.1, (Ed. Acad. R.S.R, Bucureşti, 1977).
3. E. Rădescu, N. Rădescu, C. Dumitrescu: On the Sumatory Function Associated to the Smarandache Function (Smarandache Function J. V.4-5, No. 1, (1994), 17-21).
4. F Smarandache:A Function in the Number Theory (An. Univ. Timişoara, Ser. St. Mat., Vol. XVIII, fasc.1 (1980), 79-88).

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SMARANDACHE FUNCTIONS OF THE SECOND KIND

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The Smarandache functions of the second kind are defined in [1] thus:

$$S^k : \mathbf{N}^* \to \mathbf{N}^*, \quad S^k(n) = S_n(k) \quad \text{for } n \in \mathbf{N}^*,$$

where S_n are the Smarandache functions of the first kind (see [3]).

We remark that the function S^1 has been defined in [4] by F. Smarandache because $S^1 = S$.

Let, for example, the following table with the values of S^2:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$S^2(n)$	1	4	6	6	10	6	14	12	12	10	22	8	26	14

Obviously, these functions S^k aren't monotony, aren't periodical and they have fixed points.

1. Theorem. *For $k,n \in \mathbf{N}^*$ is true $S^k(n) \le n \cdot k$.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ and $S(n) = \max\limits_{1 \le i \le t}\{S_{p_i}(\alpha_i)\} = S(p_j^{\alpha_j})$.

Because $S^k(n) = S(n^k) = \max\limits_{1 \le i \le t}\{S_{p_i}(\alpha_i k)\} = S(p_r^{\alpha_r k}) \le kS(p_r^{\alpha_r}) \le kS(p_j^{\alpha_j}) = kS(n)$ and $S(n) \le n$, [see [3]], it results:

$$(1) \qquad S^k(n) \le n \cdot k \quad \text{for every } n,k \in \mathbf{N}^*.$$

2. Theorem. *All prime numbers $p \ge 5$ are maximal points for S^k, and*

$$S^k(p) = p[k - i_p(k)], \quad \text{where} \quad 0 \le i_p(k) \le \left[\frac{k-1}{p}\right]$$

Proof. Let $p \ge 5$ be a prime number. Because $S_{p-1}(k) < S_p(k)$, $S_{p+1}(k) < S_p(k)$ [see [2]] it results that $S^k(p-1) < S^k(p)$ and $S^k(p+1) < S^k(p)$, so that $S^k(p)$ is a relative maximum value.

Obviously,

(2) $S^k(p) = S_p(k) = p[k - i_p(k)]$ with $0 \le i_p(k) \le \left[\frac{k-1}{p}\right]$.

(3) $S^k(p) = pk$ for $p \ge k$.

3. Theorem. *The numbers kp. for p prime and p>k are the fixed points of* S^k.

Proof. Let p be a prime number, $m = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ be the prime factorization of m and $p > \max\{m, k\}$. Then $p_i \alpha_i \le p_i^{\alpha_i} < p$ for $i \in \overline{1,t}$, therefore we have:

$$S^k(m \cdot p) = S[(mp)^k] = \max_{1 \le i \le t}\left\{S_{p_i^{\alpha_i}}, S_p(k)\right\} = S_p(k) = kp.$$

For $m=k$ we obtain:

$S^k(kp) = kp$ so that kp is a fixed point.

4. Theorem. *The functions* S^k *have the following properties:*

$$S^k = 0\ (n^{1+\varepsilon}), \text{ for } \varepsilon > 0$$

$$\limsup_{n \to \infty} \frac{S^k(n)}{n} = k.$$

Proof. Obviously,

$$0 \le \lim_{n \to \infty} \frac{S^k(n)}{n^{1+\varepsilon}} = \lim_{n \to \infty} \frac{S(n^k)}{n^{1+\varepsilon}} \le \lim_{n \to \infty} \frac{kS(n)}{n^{1+\varepsilon}} = k \lim_{n \to \infty} \frac{S(n)}{n^{1+\varepsilon}} = 0 \quad \text{for}$$

$$S = 0\ (n^{1+\varepsilon}), \quad \text{[see[4]]}.$$

Therefore we have $S^k = 0\ (n^{1+\varepsilon})$, and:

$$\limsup_{n \to \infty} \frac{S^k(n)}{n} = \limsup_{n \to \infty} \frac{S(n^k)}{n} = \lim_{\substack{p \to \infty \\ p \text{ prime}}} \frac{S(p^k)}{p} = k$$

5. Theorem, [see[1]]. *The Smarandache functions of the second kind standardise* $(\mathbf{N}^*, \cdot)$ *in* $(\mathbf{N}^*, \leq, +)$ *by:*

$$\Sigma_3: \ \max\left\{S^k(a), S^k(b)\right\} \leq S^k(ab) \leq S^k(a) + S^k(b)$$

and $(\mathbf{N}^*, \cdot)$ *in* $(\mathbf{N}^*, \leq, \cdot)$ *by:*

$$\Sigma_4: \ \max\left\{S^k(a), S^k(b)\right\} \leq S^k(ab) \leq S^k(a) \cdot S^k(b) \quad \text{for every} \quad a, b \in \mathbf{N}^*$$

6. Theorem. *The functions* S^k *are, generally speaking, increasing. It means that:*

$$\forall n \in \mathbf{N}^* \ \ \exists m_0 \in \mathbf{N}^* \ \textit{so that} \ \forall m \geq m_0 \ \Rightarrow \ S^k(m) \geq S^k(n)$$

Proof. The Smarandache function is generally increasing, [see [4]], it means that :

$$(3) \qquad \forall t \in \mathbf{N}^* \ \ \exists r_0(t) \in \mathbf{N}^* \ \text{so that} \ \forall r \geq r_0 \ \Rightarrow \ S(r) \geq S(t)$$

Let $t = n^k$ and $r_0 = r_0(t)$ so that $\forall r \geq r_0 \ \Rightarrow \ S(r) \geq S(n^k)$.
Let $m_0 = \left[\sqrt[k]{r_0}\right] + 1$. Obviously $m_0 \geq \sqrt[k]{r_0} \Leftrightarrow m_0^k \geq r_0$ and $m \geq m_0 \Leftrightarrow m^k \geq m_0^k$.
Because $m^k \geq m_0^k \geq r_0$ it results $S(m^k) \geq S(n^k)$ or $S^k(m) \geq S^k(n)$.
Therefore

$$\forall n \in \mathbf{N}^* \ \ \exists m_0 = \left[\sqrt[k]{r_0}\right] + 1 \quad \text{so that}$$

$$\forall m \geq m_0 \ \Rightarrow \ S^k(m) \geq S^k(n) \quad \text{where} \quad r_0 = r_0(n^k)$$

is given from (3).

7. Theorem. *The function* S^k *has its relative minimum values for every* $n = p!$, *where* p *is a prime number and* $p \geq \max\{3, k\}$.

Proof. Let $p! = p_1^{i_1} \cdot p_2^{i_2} \cdots p_m^{i_m} \cdot p$ be the canonical decomposition of $p!$, where $2 = p_1 < 3 = p_2 < \cdots < p_m < p$. Because $p!$ is divisible by $p_j^{i_j}$ it results $S(p_j^{i_j}) \leq p = S(p)$ for every $j \in \overline{1,m}$.
Obviously,

$$S^k(p!) = S[(p!)^k] = \max_{1 \leq j \leq m}\left\{S\left(p_j^{k \cdot i_j}\right), S(p^k)\right\}$$

Because $S\left(p_j^{k \cdot i_j}\right) \leq kS(p_j^{i_j}) < kS(p) = kp = S(p^k)$ for $k \leq p$, it results that we have

$$(4) \qquad S^k(p!) = S(p^k) = kp, \ \text{for} \ k \leq p$$

Let $p!-1 = q_1^{i_1} \cdot q_2^{i_2} \cdots q_t^{i_t}$ be the canonical decomposition for $p!$-1, then $q_j > p$ for $j \in \overline{1,t}$.

It follows $S(p!-1) = \max_{1 \le j \le t} \left\{ S(q_j^{i_j}) \right\} = S(q_m^{i_m})$ with $q_m > p$.

Because $S(q_m^{i_m}) > S(p) = S(p!)$ it results $S(p!-1) > S(p!)$.

Analogous it results $S(p!+1) > S(p!)$.

Obviously

$$(5) \qquad S^k(p!-1) = S\left[(p!-1)^k\right] \ge S(q_m^{k \cdot i_m}) \ge S(q_m^k) > S(p^k) = kp$$

$$(6) \qquad S^k(p!+1) = S\left[(p!+1)^k\right] > k \cdot p$$

For $p \ge \max\{3, k\}$ out of (4), (5), (6) it results that $p!$ are the relative minimum points of the functions S^k.

REFERENCES

[1] **I. Bălăcenoiu**, *Smarandache Numerical functions*, Smarandache Function Journal, vol. 4-5, no.1, (1994), p.6-13.

[2] **I. Bălăcenoiu**, *The monotony of Smarandache functions of first kind*, Smarandache Function Journal, vol.6, 1995.

[3] **I. Bălăcenoiu, V. Seleacu**, *Some properties of Smarandache functions of the type I*, Smarandache Function Journal, vol.6, 1995.

[4] **F. Smarandache**, *A function in the Number Theory*. An.Univ.Timişoara, seria şt.mat. Vol.XVIII, fasc. 1, p.79-88, 1980.

THE PROBLEM OF LIPSCHITZ CONDITION

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In our paper we prove that the Smarandache function S does not verify the Lipschitz condition, giving an answer to a problem proposed in [2] and we investigate also the possibility that some other functions, which involve the function S, verify or not verify the Lipschitz condition.

Proposition 1 *The function $\{n \to S(n)\}$ does not verify the Lipschitz condition, where $S(n)$ is the smallest integer m such that $m!$ is divisible by n. (S is called the Smarandache function.)*

Proof. A function $f : M \subseteq R \to R$ is Lipschitz iff the following condition holds:

$$(\exists)K > 0, (\forall)x, y \in M \Rightarrow | f(x) - f(y) | \leq K \mid x - y \mid$$

(K is called a Lipschitz constant).

We have to prove that for every real $K > 0$ there existe $x, y \in N^*$ such that $| f(x) - f(y) | > K \mid x - y \mid$.

Let $K > 0$ be a given real number. Let $x = p > 3K + 2$ be a prime number and consider $y = p + 1$ which is a composite number, beeig even. Since $x = p$ is a prime number we have $S(p) = p$. Using [1] we have $\max\limits_{n \in N^*, n \neq 4} \{S(n)/n\} = 2/3$, then $\frac{S(y)}{y} = \frac{S(p+1)}{p+1} \leq \frac{2}{3}$ which implies that $S(p + 1) \leq \frac{2}{3}(p + 1) < p = S(p)$. We have

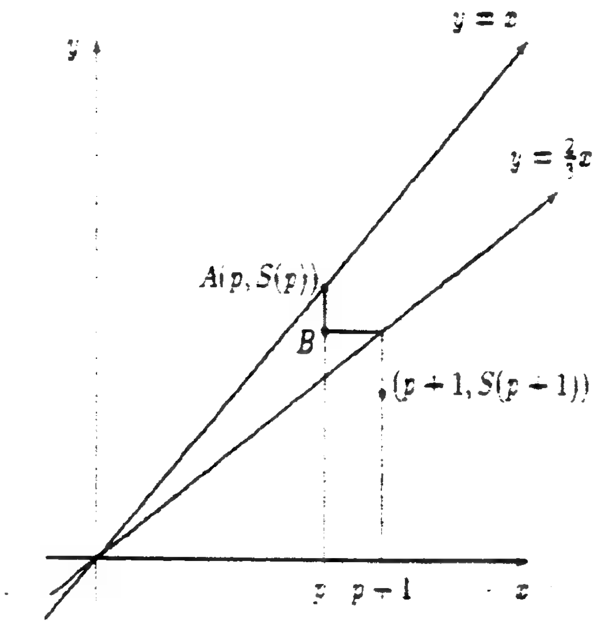
$$|S(p) - S(p + 1)| = p - S(p + 1) \geq p - \frac{2}{3}(p + 1) > \frac{3K + 2 - 2}{3} = K$$

∎

Remark 1. The ideea of the proof is based on the following observations:

If p is a prime number, then $S(p) = p$, thus the point $(p, S(p))$ belongs to the line of equation $y = x$;

If q is a composite integer, $q \neq 4$, then $\frac{S(q)}{q} \leq \frac{2}{3}$ which means that the point $(q, S(q))$ is under the graphic of the line of equation $y = \frac{2}{3}x$ and above the axe $\overrightarrow{Ox}$.



Thus, for every consecutive integer numbers x, y where $x = p$ is a prime number and $y = p+1$, the leught AB can be made as great as we need, for x, y sufficiently great.

Remark 2. In fact we have proved that the function $f : N^* \to N$ defined by $f(n) = |S(n) - S(n-1)|$ is unbounded, which imply that the Smarandache's function is not Lipschitz.

In the sequel we study the Lipschitz condition for other functions which involve the Smarandache's function.

Proposition 2 *The function* $S_1 : N \setminus \{0,1\} \to N$, $S_1(n) = \frac{1}{S(n)}$ *verify the Lipschitz condition.*

Proof. For every $x \geq 2$ we have $S(x) \geq 2$, therefore $0 < \frac{1}{S(x)} \leq \frac{1}{2}$. If we take $x \neq y$ in $N \setminus \{0,1\}$, we have

$$\left| \frac{1}{S(x)} - \frac{1}{S(y)} \right| \leq \frac{1}{2} \leq \frac{1}{2}|x-y| .$$

For $x = y$ we have an equality in the relation above, therefore S_1 is a function which verify the Lipschitz condition with $K = \frac{1}{2}$ and more, it is a contractant function.

Remark 3. In [2] it is proved that $\sum_{n\geq 1} \frac{1}{S(n)}$ is divergent .

Proposition 3 *The function $S_2 : N^* \to N$, $S_2(n) = \frac{S(n)}{n}$ verify the Lipschitz condition.*

Proof. For every $x, y \in N$, $1 < x < y$ we have $x = n$ and $y = n + m$ where $m \in N^*$. In [2] is proved that
$$\frac{1}{(n-1)!} \leq \frac{S(n)}{n} \leq 1, \ (\forall)n \in N \setminus \{0,1\}.$$
Using this we have
$$\left|\frac{S(x)}{x} - \frac{S(y)}{y}\right| = \left|\frac{S(n)}{n} - \frac{S(n+m)}{n+m}\right| \leq 1 - \frac{1}{(n+m-1)!} < 1 \leq |x - y|$$
therefore
$$\left|\frac{S(x)}{x} - \frac{S(y)}{y}\right| \leq |x - y|$$
for x and y as above. For $x = y$ we have an equality in the relation above. It follows that S_2 is verify the Lipschitz condition with $K = 1$.

■

Remark 4. Using the proof of *Proposition 5* proved below, it can be shown that the Lipschitz constant $K = 1$ is the best possible. Indeed, take $x = n = p - 1$, $m = 1$ and therefore $y = p$ (with the notations from the proof of *Proposition 3*), with p a primenumber. From the proof of *Proposition 5*, there is a subsequence of prime numbers $(p_{n_k})_{k\geq 1}$ such that $\frac{S(p_{n_k}-1)}{p_{n_k}-1} \xrightarrow{k\to\infty} 0$. For $k \geq 1$ we have, for a Lipschitz constant K of S_2 :
$$K \geq \left|\frac{S(p_{n_k})}{p_{n_k}} - \frac{S(p_{n_k}-1)}{p_{n_k}-1}\right| = 1 - \frac{S(p_{n_k}-1)}{p_{n_k}-1} \xrightarrow{k\to\infty} 1$$
Thus, $K \geq 1$

Proposition 4 *The function $S_3 : N \setminus \{0, 1\} \to N$, $S_3(n) = \frac{n}{S(n)}$ does not verify the Lipschitz condition.*

Proof. (Compare with the proof of *Proposition 1*.)

We have to prove that for every real $K > 0$ there existe $x, y \in N^*$ such that $|S_3(x) - S_3(y)| > K|x - y|$.

Let $K > 0$ be a given real number, $x = p$ be a prime number and $y = x - 1$. Using the *Proposition 5* proved below, which asserts that the sequence $\left\{\frac{p_n - 1}{S(p_n - 1)}\right\}_{n \geq 2}$ is unbunded (where $\{p\}_{n\geq 1}$ is the prime numbers sequence), we have, for a prime number p such that $\frac{p-1}{S(p-1)} > K + 1$:

$$\left|\frac{x}{S(x)} - \frac{y}{S(y)}\right| = \left|\frac{p}{S(p)} - \frac{p-1}{S(p-1)}\right| = \frac{p-1}{S(p-1)} - 1 > K + 1 - 1 = K = K|x - y|$$

■

Proposition 5 *If $\{p_n\}_{n\geq 1}$ is the prime numbers sequence, then the sequence $\left\{\frac{p_n - 1}{S(p_n - 1)}\right\}_{n\geq 2}$ is unbounded.*

Proof. Denote $q_n = p_n - 1$ and let r_n be the number of the distinct prime numbers which appear in the prime factor decomposition of q_n, for $n \geq 2$. We show below that $\{r_n\}_{n\geq 2}$ is an unbounded sequence.

For a fixed $k \in N^*$, consider $\pi_k \stackrel{def}{=} p_1 \cdots p_k$ and the arithmetic progression $\{1 + \pi_k \cdot m\}_{m\geq 1}$. From the Dirichlet Theorem [3, pg.194], it follows that this sequence contains a subsequence $\{1 + \pi_k \cdot m_l\}_{l\geq 1}$ of prime numbers: $p_{n_l} = 1 + \pi_k \cdot m_l$, therefore $\pi_k \cdot m_l = p_{n_l} - 1 = q_{n_l}$ which implies that $r_{n_l} \geq k$. It shows that the sequence $\{r_n\}_{n\geq 2}$ is an unbounded sequence.

If $q_n = \prod_{i=1}^{r_n} p_{\beta_i}^{\alpha_i}$ then it is known (see [4]) that:

$$S(q_n) = \max_{i=\overline{1,r_n}} \left\{S\left(p_{\beta_i}^{\alpha_i}\right)\right\} = S\left(p_{\beta_j}^{\alpha_j}\right) \leq \alpha_j p_{\beta_j}$$

thus

$$\frac{q_n}{S(q_n)} = \frac{\prod_{i=1}^{r_n} p_{\beta_i}^{\alpha_i}}{S\left(p_{\beta_j}^{\alpha_j}\right)} \geq \left(\prod_{i=1, i\neq j}^{r_n} p_{\beta_i}^{\alpha_i}\right) \frac{p_{\beta_j}^{\alpha_j - 1}}{\alpha_j}. \tag{1}$$

We have:

$$u_j = \frac{p_{\beta_j}^{\alpha_j - 1}}{\alpha_j} \geq 2 \tag{2}$$

Indeed, if $\alpha_j = 1$, then $u_j = 1$. If $\alpha_j > 1$, then

$$u_j \geq \frac{(p_j - 1)(\alpha_j - 1)}{\alpha_j} \geq \frac{p_j - 1}{2} \geq \frac{1}{2}$$

But $v_n = \prod_{i=1, i \neq j}^{r_n} p_{\beta_i}^{\alpha_i}$ has $r_n - 1$ prime factors and $\{r_n\}_{n \geq 2}$ is unbounded, then it follows that $\{v_n\}_{n \geq 2}$ is unbounded. Using this, (1) and (2), it follows that the sequence $\left\{ \frac{q_n}{S(q_n)} \right\}_{n \geq 2}$ is unbounded.

∎

Remark 5. Using the same ideea, the *Proposition 5* is true in a more general form:

For $a \in Z$, the sequence $\left\{ \frac{p_n + a}{S(p_n + a)} \right\}_{p_n + a \geq 2}$ is unbounded, where $\{p_n\}_{n \geq 1}$ is the prime numbers sequence.

References

[1] T. Yau *A problem of maximum*, Smarandache Function Journal, vol.4-5, No.1, September 1994, pg.45.

[2] F. Smarandache *An infinity of unsolved problems concerning a function in the number theory*, Smarandache Function Journal, vol.1, No.1, December 1994, pg.18-66.

[3] C Popovici *Teoria numerelor*, Editura didactică şi pedagogică, Bucureşti, 1973.

[4] Pål Gronås *A proof of the non-existence of SAMMA*, Smarandache Function Journal, vol.4-5, No.1, September 1994, pg.22-23.

A BRIEF HISTORY OF THE "SMARANDACHE FUNCTION" (III)

by Dr. Constantin Dumitrescu

ADDENDA (III) :
New References concerninig this function (got by the editorial board after August 1, 1994):

{ See the previous two issues of the journal for the first and second parts of this article }

****

[95] The journal was indexed by the <Mathematical Reviews>, Ann Arbor, MI, 94c, March 1994, XXI;

[96] David E. Zitarelli, review of "A brief history of the <Smarandache Function>", in <HISTORIA MATHEMATICA>, Academic Press, Inc., Harcourt Brace & Co., San Diego, New York, Boston, London, Sydney, Tokyo; Vol. 21, No. 1, February 1994, 102; #21.1.42;
and in <HISTORIA MATHEMATICA>, Vol. 21, No. 2, May 1994, 229; #21.2.28, #21.2.29;

[97] Carol Moore, Arizona State University Library, Letter to C. Dumitrescu and V. Seleacu concerning the Smarandache Function Archives, April 20, 1994;

[98] T. Yau, "Teaching the Smarandache Function to the American Competition Students", abstract, Department of Mathematics, University of Oregon, 1994;
Letter from Richard M. Koch, 6/14/94;

[99] George Fernandez, Paradise Valley Community College, "An inequation concerning the Smarandache Function", to the International Congress of Mathematicians (ICM 94), Zürich, 3-11 August 1994;

[100] George Mițin vărieșescu, Sydney, Australia, abstract in "Orizonturi Albastre / Poeți Români în Exil", Cogito Publishing House, Oradea, 1993, 89-90;

[101] Paula Shanks, <Mathematical Reviews>, Letter to R. Muller, December 6, 1993;

[102] Harold W. Billings, Director of General Libraries, The University of Texas at Austin, "The Florentin Smarandche Papers (1978-1994)" Special Collection, Archives of American Mathematics, Center for American History, SRH 2.109, TX 78713, tel. (512) 495-4129, five linear feet;

[103] M. Andrei, C. Dumitrescu, V. Seleacu, L. TuȚescu, St. Zanfir, "Some remarks on the Smarandache Function", in <Bulletin of Pure and Applied Sciences>, Editor Prof. M. N. Gopalan, Indian Institute of Technology, Bombay, India, Vol. 13E (No. 2), 1995;

[104] I. Rotaru, "Cine este Florentin Smarandache ?", preface for "Fugit... jurnal de lagăr", p. 5, Ed. Tempus, Bucharest, 1994;

[105] Geo Stroe, postface for "Fugit... jurnal de lagar",

cover IV, Ed. Tempus, Bucharest, 1994;

[106] Peter Bundschuh, Köln, "Auswertung der eingesandten Lösungen", in <Elemente der Mathematik>, Switzerland, Vol. 49, No. 3, 1994, 127-8;

[107] Gh. Tomozei, "Funcţia Smarandache", preface to <Exist împotriva mea>, pre-paradoxist poetry by F. Smarandache, Ed. Macarie, Târgovişte, 1994, pp. 5-9;
also in <Literatorul>, Bucharest, Nr. 42 (159), 14-21 October 1994, p.6;

[108] Khalid Khan, London School of Economics, "Letter to the Editor / The Smarandache function", in <Mathematical Spectrum>, Vol. 27, No. 1, 1994/5, 20-1;

[109] Pål Grønås, Stjordal, Norway, "Letter to the Editor / The Smarandache function", in <Mathematical Spectrum>, Vol. 27, No. 1, 1994/5, 21;

[110] Khalid Khan, London School of Economics, Solution to Problem 26.8, in <Mathematical Spectrum>, Vol. 27, No. 1, 1994/5, 22;
also solved by David Johansen and Polly Show, Dame Allan's Girls' School, Newcastle upon Tyne, U. K.;

[119] Jane Friedman, "Smarandache in Reverse" / solution to problem B-740, in <The Fibonacci Quarterly>, USA, November 1994, pp. 468-9;

[120] A. Stiuparu, Problem H-490, in <The Fibonacci Quarterly>, Vol. 32, No. 5, November 1994, p.473;

[121] Dumitru Ichim, Cronici, in <Cuvântul Românesc>, Hamilton, Ontario, Canada, Anul 20, Nr. 221, November 1994, p.12;

[122] Mihaly Bencze, Open Question: OQ 6, in <Octogon>, Braşov, Vol. 2, No. 1, April 1994, p.34;

[123] Pr. R. Halleux, rédacteur en chef, <Archives Internationales d'Histoire des Sciences>, Université de Liège, Belgique, Lettre vers R. Muller, le 14 novembre 1994;

[124] Florentin Gh. Smarandache, "An Infinity of Unsolved Problems concerning a Function in the Number Theory", abstract in <Proceedings of the International Congress of Mathematicians>, Section 3: Number Theory, University of Berkeley, CA, USA, 1986;

[125] F. Smarandache, Problem 7856, in <Gamma>, Braşov, Anul X, No. 3-4 (31-31), February 1988, p.77;

[126] Marian Mirescu, "Catedrala Funcţiei Smarandache" (drawing), in <Abracadabra>, Salinas, CA, December 1994, p. 20;

[127] A. D. Rachieru, " 'Avalanşa' Smarandache", in <Banatul>, Timişoara, Nr. 4, 1994;

[128] Gh. Suciu, "Spre America - Via Istambul", in <Minerva>, Bistriţa-Năsăud, Anul V, No. 39-40, p.10, October - November 1994;

[129] Ion Radu Zăgreanu, " 'Exist împotriva mea' ", in <Minerva>, Bistriţa-Năsăud, Anul V, No. 39-40, p.10, October - November 1994;

[130] R. Muller, editor of "Unsolved Problems related to Smarandache Function", Number Theory Publ. Co., Phoenix,

1993;
reviewed in <Mathematical Reviews>, Ann Arbor, 94m:11005, 11-06;

[131] Gh. Stroe, "Smarandache Function", in <Tempus>, Bucharest, Anul III. Nr. 2(5), November 1994, p.4;

[132] Dr. Dumitru Acu, University of Sibiu, "Funcţia Smarandache...", in <Abracadabra>, Salinas, CA, January 1995, No. 27, Anul III, p.20;

[133] Lucian Tuţescu, "...funcţia Smarandache...", in <Abracadabra>, Salinas, CA, January 1995, No. 27, Anul III, p.20;

[134] Constantin M. Popa, "Funcţia...", in <Abracadabra>, Salinas, CA, January 1995, No. 27, Anul III, p.20;

[135] Prof. M. N. Gopalan, Editor of <Bulletin of Pure & Applied Sciences>, Bombay, India, Letter to M. Andrei, December 26, 1994;

[136] Dr. Peter L. Renz, Academic Press, Cambridge, Massachusetts, Letter to R. Muller, January 11, 1995;

[137] Charles Ashbacher, review of the "Smarandache function Journal", in <Journal of Recreational Mathematics>, USA, Vol. 26(2), pp. 138-9, 1994;

[138] N. J. A. Sloane, S. Plouffe, B. Salvy, "The Encyclopaedia of Integer Sequences", Academic Press, 1995, M0453 N0167;
also online: SUPERSEEKER@RESEARCH.ATT.COM ;

[139] Editors of <Mathematical Reviews>, review of the book "Unsolved Problems related to Smarandache Function" by F. Smarandache, 94m:11005;

[140] Jean-Marie De Koninck, Quebec, review of the paper "A function in the number theory" by F. Smarandache, in <Mathematical Reviews>, 94m:11007, p.6940;

[141] Jean-Marie De Koninck, Quebec, review of the paper "Some linear equations involving a function in the number theory" of F. Smarandache, in <Mathematical Reviews>, 94m:11008, p.6940;

[142] Armel Mercier, review of the paper "An infinity of unsolved problems concerning a function in the number theory" of F. Smarandache, in <Mathematical Reviews>, 94m:11010, p.6940;

[143] Armel Mercier, review of the paper "Solving problems by using a function in the mnumber theory" of F. Smarandache, in <Mathematical Reviews>, 94m:11011, p.6941;

[144] I. M. Radu, Bucharest, Letter to the Editor ("The Smarandache function"), in <Mathematical Spectrum>, Sheffield University, UK, Vol. 27, No. 2, p. 43, 1994/5;

[145] Paul Erdos, Hungarian Academy of Sciences, Letter to the Editor ("The Smarandache function inter alia"), in <Mathematical Spectrum>, Vol. 27, No. 2, pp. 43-4, 1994/5;

[146] I. Soare, "Un scriitor al paradoxurilor: Florentin Smarandache", 114 pages, Ed. Almarom, Rm. Vâlcea, Romania, p. 67, 1994;

[147] Dr. C. Dumitrescu, "Functia Smarandache", in <Foaie

Matematică>, Chişinău, Republic of Moldova, Nr. 3, 1995, p. 43;

[148] A. Stuparu, D. W. Sharpe, Problem 1, in <Foaie Matematica>, Chisinau, Republic of Moldova, Nr. 3, 1995, p. 43;

[149] Pedro Melendez, Problem 2, in <Foaie Matematica>, Chisinau, Republic of Moldova, Nr. 3, 1995, p. 43;

[150] Ken Tauscher, Problem 3, in <Foaie Matematica>, Chisinau, Republic of Moldova, Nr. 3, 1995, p. 43;

[151] T. Yau, Problem 4, in <Foaie Matematica>, Chisinau, Republic of Moldova, Nr. 3, 1995, p. 43;

[152] T. Popescu, "Estetica Paradoxismului", (see Introduction), 150 pp., 1995;

[153] N. J. A. Sloane & S. Plouffe, "The Encyclopedia of Integer Sequences", Academic Press, San Diego, New York, Boston, London, Sydney, Tokyo, Toronto, 1995;
also online, email: superseeker@research.att.com
(SUPERSEEKER by N. J. A. Sloane, S. Plouffe, B. Salvy, ATT Bell Labs, Murray Hill, NJ 07974, USA); presented as:

"SMARANDACHE NUMBERS": S(n), for n = 1, 2, 3, ..., [M0453],

and

"SMARANDACHE QUOTIENTS": for each integer n > 0, find the smallest k such that nk is a factorial; [M1669];

and

"SMARANDACHE DOUBLE FACTORIALS": F(n) is the smallest integer such that F(n)!! is divisible by n; [A7922] in the electronic version.

PROPOSED PROBLEM

by Thomas Martin

Let $\eta: Z^* \to N$ Smarandache Function: $\eta(m)$ is the smallest integer n such that $n!$ is divisible by m.

a) Prove that for any number $k \in R_*$ there exist a series $\{p_i\}_i$ of positive integer numbers such that :

$$L = \lim_{i\to\infty} \frac{p_i}{\eta(p_i)} > k$$

b) Does $L = \lim_{n\to\infty} \frac{m}{\eta(m)}$ diverge to $+\infty$.

Solution:

a) Let p_j be a prime number greater than k. Index j is fixed. We construct $p_i = p_j p_{j+i}$, for $i = 1,2,3\ldots$.

Lemma 1. If $u < v$ are prime numbers, then $\eta(uv) = v$.

Of course $v! = 1\cdot 2\cdot\ldots\ldots\cdot u\cdot\ldots\cdot v = \mathcal{M}_v = \mathcal{M}_u$.

Hence $\eta(p_i) = p_{j+i}$, for any $i = 1,2,3\ldots$ where p_{j+i} is the $j+i^{th}$ prime number. Then $L = p_j > k$.

b) Because there exists an infinity of primes : $p_j, p_{j+1}, \ldots,$ greater than k, we find an infinity of limits for each $\{p_i(j)\}_i$ series, i.e. $L = p_{j+1}$ or $L = p_{j+2}$ etc.

Therefore $L = \lim_{m\to\infty} \frac{m}{\eta(m)}$ does not exist!

Reference:

R. Muller, "Smarandache Function Journal", Vol. 1. No. 1, 1990.

PROPOSED PROBLEM

by J. Thompson

Calculate:

$$\lim_{n\to\infty}\left(1+\sum_{k=2}^{n}\frac{1}{\eta(k)}-\log\eta(n)\right)$$

where $\eta(n)$ is Smarandache Function : the smallest integer m , such that $m!$ is divisible by n .

Solution:

We know that $\left(\sum_{k=1}^{n} 1/k-\log n\right)$ converges to e for $n\to\infty$.

It's easy to show that for $k\geq 2$, $\eta(k)\leq k$. More, for k a composite number ≥ 10, $\eta(k)\leq k/2$. Also, if $p>4$ then : $\eta(p)=p$ if and only if p is prime.

$$\sum_{k=10}^{n}\frac{1}{\eta(k)}-\log\eta(n)\geq\left(\sum_{k=10}^{n}\frac{1}{k}-\log n\right)+\sum_{\substack{k=10\\ k\neq prime}}\frac{1}{k}\xrightarrow{n\to\infty} e+\infty=\infty$$

because for any prime number p there exists a composite number $p-1$ such that $\frac{1}{p-1}>\frac{1}{p}$ thus :

$$\sum_{\substack{k=10\\ k\neq prime}}\frac{1}{k}=\frac{1}{10}+\frac{1}{12}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\frac{1}{18}+\ldots+\frac{1}{n}>\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\ldots+\frac{1}{p(n)}\xrightarrow{n\to\infty}\infty$$

where $p(n)$ is the greatest prime number less that n .

We took out the first nine terms of that series, the limit of course didn't chance.

Reference:

Smarandache F., " A function in the number theory", <Analele Univ. Timisoara>, fasc. 2, Vol. XVII,pp. 163-8, 1979;
see Mathematical Review: 82a:03012.

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PROPOSED PROBLEM OF NUMBER THEORY

BY PROF. KEN TAUSCHER

Let N be a positif integer. Let η be the function that associates to any non-null integer P the smallest number Q such find the minimum value of K from which $\eta(R) > N$ for any $R > K$.

Solution:
Lemma: For any $X > Y!$ we have $\eta(X) > Y$.
Proof by reductio ad absurdum:
If $\eta(X) = A \leq Y$, then $A! \leq Y! < X$, whence $A!$ may not be divisible by X.

Reference:

Thomas Martin, Aufgabe 1075, "Elemente der mathematik", vol. 49, No. 3, 1993.

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A GENERALIZATION OF A PROBLEM OF STUPARU

by L. Seagull, Glendale Community College

Let n be a composite integer $>= 48$. Prove that between n and $S(n)$ there exist at least 5 prime numbers.

Solution:

T. Yau proved that Smarandache function has the following property:

$S(n) <= n/2$ for any composite number $n >= 10$,

because:

if $n = pq$, with $p < q$ and $(p, q) = 1$, then:

$S(n) = \max \{S(p), S(q)\} = S(q) <= q = n/p <= n/2$;

if $n = p^r$, with p prime and r integer $>= 2$, then:

$S(n) <= pr <= (p^r)/2 = n/2$.

(Inequation $pr <= (p^r)/2$ doesn't hold:

for $p = 2$ and $r = 2, 3$;

as well as for $p = 3$ and $r = 2$;

but in either case $n = p^r$ is less than 10.

For $p = 2$ and $r = 4$, we have $8 <= 16/2$;

therefore for $p = 2$ and $r >= 5$, inequality holds because the right side is exponentially increasing while the left side is only linearly increasing,

i.e. $2r <= (2^r)/2$ for $r >= 4$ (1)

Similarly for $p = 3$ and $r >= 3$,

i.e. $3r <= (3^r)/2$ for $r >= 3$. (2)

Both of these inequalities can be easily proved by induction.

For $p = 5$ and $r = 2$, we have $10 <= 25/2$;

and of course for $r >= 3$ inequality $5r <= (5^r)/2$ will hold.

If $p >= 7$ and $r = 2$, then $p2 <= (p^2)/2$,

which can be also proved by induction.)

Stuparu proved, using Bertrand/Tchebychev postulate/theorem, that there exists at least one prime between n and $n/2$ {i.e. between n and $S(n)$}.

But we improve this if we apply Breusch's Theorem,

which says that between n and $(9/8)n$ there exists at least one prime.

Therefore, between n and $2n$ there exist at least 5 primes,

because $(9/8)^5 = 1.802032470703125... < 2$,

while $(9/8)^6 = 2.027286529541016... > 2$.

References:

I. M. Radu, "Mathematical Spectrum", Vol. 27, No. 2, p. 43, 1994/5.

D. W. Sharpe, Letters to the Author, 24 February & 16 March, 1995.

AN IMPORTANT FORMULA TO CALCULATE THE NUMBER OF PRIMES LESS THAN X

by L. Seagull, Glendale Community College

If x >= 4, then:

$$\pi(x) = \sum_{k=2}^{x} \left\lfloor \frac{S(k)}{k} \right\rfloor - 1$$

where S(k) is the Smarandache Function: the smallest integer such that S(k)! is divisible by k, and $\left\lfloor a \right\rfloor$

means the integer part of a.

Proof:

Knowing the Smarandache Function has the property that if p > 4 then S(p) = p if only if p is prime,
and S(k) <= k for any k,
and S(4) = 4 (the only exception from the first rule),
we easily find an exact formula for the number of primes less or equal than x.

CONTENTS

A collection of papers concerning Smarandache type functions, numbers, sequences, integer algorithms, paradoxes, experimental geometries, algebraic structures, neutrosophic probability, set, and logic, etc.

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